LIMIT

EXERCISE – I HINT & SOLUTION

Sol.1 C

$\lim_{x\to 1} (1 - x + [x - 1] + [1 - x])$

Let f(x) = 1 – x + [x – 1] + [1 – x]

= 1 – x + [x] + [–x]

Now, LHL at x = 1

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \{1 - x + [x] + [-x]\}$

$\Rightarrow \lim_{h\to 0} \{1 - (1 - h) + [1 - h] + [-(1 - h)]\}$

$\Rightarrow \lim_{h\to 0} \{h + [1 - h] + [-1 + h]\}$

$\Rightarrow 0 + 0 + -1 = -1$

RHL at x = 1

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \{1 - x + [x] + [-x]\}$

$\Rightarrow \lim_{h\to 0} \{1 - (1 + h) + [1 + h] + [-1 - h]\}$

$\Rightarrow 0 + 1 + (-2) = -1$ so, $\lim_{x\to 1} f(x) = -1$

Sol.2 D

$\lim_{x\to\infty} \sec^{-1}\left(\frac{x}{x+1}\right)$ Put $x = \frac{1}{y}$

$= \lim_{y\to 0} \sec^{-1}\left(\frac{1}{y+1}\right) = \lim_{y\to 0} \cos^{-1}(y + 1)$

Now, LHL at y = 0

$\lim_{y\to 0^-} \cos^{-1}(y + 1) = \lim_{h\to 0} \cos^{-1}(1 - h) = 0$

RHL at y = 0

$\lim_{y\to 0^+} \cos^{-1}(y+1) = \lim_{h\to 0} \cos^{-1}(1+h) \Rightarrow$ Not possible

so, Limit does not exists.

Sol.3 D

$\lim_{x\to 2} \frac{\sin\left(e^{x-2} - 1\right)}{\ell n\,(x-1)}$

$= \lim_{x\to 2} \frac{\sin\,(e^{x-2} - 1)}{(e^{x-2} - 1)} \times \frac{(e^{x-2} - 1)}{\ell n\,(x-1)}$

$= \lim_{x\to 2} \frac{e^{(x-2)} - 1}{(x-2)} \times \frac{(x-2)}{\ell n\,(1 + (x-2))} = 1 \times 1 = 1$

Aliter : → we can do the same question by L – Hospital rule also,

$\lim_{x\to 2} \frac{\sin\,(e^{(x-2)} - 1)}{\ell n\,(x-1)} = \lim_{x\to 2} \frac{\cos(e^{(x-2)} - 1) \times e^{(x-2)}}{\frac{1}{(x-1)}}$

$= \lim_{x\to 2} (x - 1) \,.\, e^{(x-2)} \,.\, \cos\,(e^{x-2} - 1) = 1$

Sol.4 B

$\lim_{x\to -\infty} \frac{x^2 \sin\left(\frac{1}{x}\right)}{\sqrt{9x^2 + x + 1}}$

Put $x = -\frac{1}{y}$, so limit changes to $y \to 0$

$\lim_{y\to 0} \frac{-\sin y}{y^2\sqrt{\frac{9}{y^2} - \frac{1}{y} + 1}} = \lim_{y\to 0} \frac{-\sin y}{y\sqrt{y^2 - y + 9}}$

$= -\frac{1}{\sqrt{9}} = -\frac{1}{3}$

Sol.5 D

$\lim_{x\to 0} \frac{\sin\,(\ell n(1+x))}{\ell n\,(1 + \sin x)}$

$= \lim_{x\to 0} \frac{\sin(\ell n(1+x))}{\ell n\,(1+x)} \times \frac{\ell n\,(1+x)}{x} \times \frac{\sin x}{\ell n\,(1+\sin x)}$

= 1 . 1 . 1 = 1

Sol.6 C

$\lim_{x\to \frac{\pi}{2}} \left[\frac{x - \frac{\pi}{2}}{\cos x}\right]$ Let $x = \left(\frac{\pi}{2} + h\right)$

$= \lim_{h\to 0} \left[\frac{\frac{\pi}{2} + h - \frac{\pi}{2}}{\cos\left(\frac{\pi}{2} + h\right)}\right] = \lim_{h\to 0} \left[\frac{h}{-\sinh}\right] = -2$

Sol.7 C

$\lim_{x\to \frac{\pi}{2}} \tan^2 x\left(\sqrt{2\sin^2 x + 3\sin x + 4} - \sqrt{\sin^2 x + 6\sin x + 2}\right)$

$$= \lim_{x\to\frac{\pi}{2}} \tan^2 x\left\{\frac{2\sin^2 x+3\sin x+4-\sin^2 x-6\sin x-2}{\sqrt{2\sin^2 x+3\sin x+4}+\sqrt{\sin^2 x+6\sin x+2}}\right\}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\sin^2 x}{\cos^2 x}\times\frac{(\sin^2 x-3\sin x+2)}{(\sqrt{2+3+4}+\sqrt{1+6+2})}$$

$$= \frac{1}{6}\times\lim_{x\to\frac{\pi}{2}}\frac{(\sin x-2)(\sin x-1)}{(1-\sin x)(1+\sin x)} = \frac{1}{12}$$

Sol.8 C

$$\lim_{x\to\infty}\frac{x^3\sin\left(\frac{1}{x}\right)+x+1}{x^2+x+1}$$

$$\lim_{x\to\infty}\frac{x^2\left\{\frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}+\frac{1}{x}+\frac{1}{x^2}\right\}}{x^2\left\{1+\frac{1}{x}+\frac{1}{x^2}\right\}} = \frac{1}{1} = 1$$

Sol.9 A

$$\lim_{x\to\infty}\frac{-3n+(-1)^n}{4n-(-1)^n} = \lim_{x\to\infty}\frac{-3+\frac{(-1)^n}{n}}{4-\frac{(-1)^n}{n}}$$

$$= \lim_{x\to\infty}\frac{-3+\frac{(-1\text{ or }1)}{\infty}}{4-\frac{(-1\text{ or }1)}{\infty}} = -\frac{3}{4}$$

Sol.10 C

$$\lim_{\theta\to0}\left(\left[\frac{n\sin\theta}{\theta}\right]+\left[\frac{n\tan\theta}{\theta}\right]\right)$$

As we know,

$$\frac{\sin\theta}{\theta}<1 \Rightarrow \frac{n\sin\theta}{\theta}<n \Rightarrow \left[\frac{n\sin\theta}{\theta}\right]=(n-1)...(1)$$

Also, $\frac{\tan\theta}{\theta}>1 \Rightarrow \frac{n\tan\theta}{\theta}>n \Rightarrow \left[\frac{n\tan\theta}{\theta}\right]=n...(2)$

Now, $\lim_{\theta\to0}\left(\left[\frac{n\sin\theta}{\theta}\right]+\left[\frac{n\tan\theta}{\theta}\right]\right)=n-1+n=(2n-1)$

Sol.11 B

$$\lim_{n\to\infty} n\cos\left(\frac{\pi}{4n}\right)\sin\left(\frac{\pi}{4n}\right)$$

$$= \lim_{n\to\infty} n\,.\,\cos\left(\frac{\pi}{4n}\right)\,.\,\frac{\sin\left(\frac{\pi}{4n}\right)}{\left(\frac{\pi}{4n}\right)}\times\frac{\pi}{4n} = \frac{\pi}{4}$$

Sol.12 C

$$\lim_{x\to0}\left[\frac{\sin[x-3]}{[x-3]}\right]$$

LHL at x = 0

$$\lim_{h\to0}\left[\frac{\sin[-3-h]}{[-3-h]}\right] = \lim_{h\to0}\left[\frac{\sin(-4)}{(-4)}\right] = -1$$

RHL at x = 0

$$\lim_{h\to0}\left[\frac{\sin[3+h]}{[3+h]}\right] = \lim_{h\to0}\left[\frac{\sin 3}{3}\right] = 1$$

since LHL $\neq$ RHL hence limit does not exist.

Sol.13 A

$\lim_{x\to\infty}\left(\frac{x+2}{x-2}\right)^{(x+1)}$ It is of the form 1^∞, so

$$\ell = e^{\lim_{x\to\infty}(x+1)\left[\frac{x+2}{x-2}-1\right]} = e^{\lim_{x\to\infty}\left(\frac{x+1}{x-2}\right)[x+2-x+2]}$$

$$= e^{\lim_{x\to\infty}\left(\frac{1+1/x}{1-2/x}\right).4} = e^4$$

Sol.14 A

$\lim_{x\to0^+}\left(1+\tan^2\sqrt{x}\right)^{\frac{5}{x}}$ It is of the form 1^∞, so

$$\ell = e^{\lim_{x\to0^+}\frac{5}{x}\left(1+\tan^2\sqrt{x}-1\right)} = e^{\lim_{x\to0^+}\frac{5}{x}\left(\tan^2\sqrt{x}\right)}$$

$$= e^{\lim_{h\to0}\frac{5}{h}\left(\tan^2\sqrt{h}\right)} = e^{5\lim_{h\to0}\left(\frac{\tan^2\sqrt{h}}{\left(\sqrt{h}\right)^2}\right)} = e^5$$

Sol.15 B

$$\lim_{x\to\frac{\pi}{4}}(1+[x])^{1/\ell n\tan x}$$

After putting limit [x] becomes zero, so base is dot one hence $1^\infty = 1$

Sol.16 C

$$\lim_{x\to\infty}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x$$ It is of the form 1^∞, so,

$$\Rightarrow \ell = e^{\lim_{x\to\infty} x\left[\frac{x^2-2x+1}{x^2-4x+2}-1\right]} = e^{\lim_{x\to\infty}\left(\frac{2x^2-x}{x^2-4x+2}\right)}$$

$$= e^{\lim_{x\to\infty}\left(\frac{2-1/x}{1-4/x+2/x^2}\right)} = e^2$$

Sol.17 C

$$\lim_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$$ It is of the form 1^∞,

$$\ell = e^{\lim_{x\to a}\tan\frac{\pi x}{2a}\left(2-\frac{a}{x}-1\right)} = e^{\lim_{x\to a}\tan\frac{\pi x}{2a}\left(\frac{x-a}{x}\right)}$$

Put x = a + h,

$$\ell = e^{\lim_{h\to 0}\tan\left(\frac{\pi}{2}+\frac{\pi h}{2a}\right).\left(\frac{h}{a+h}\right)} = e^{-\lim_{h\to 0}\cot\left(\frac{\pi h}{2a}\right).\left(\frac{h}{a+h}\right)}$$

$$= e^{-\lim_{h\to 0}\frac{1}{\tan\left(\frac{\pi h}{2a}\right)}\times(\pi h/2a)\times\frac{2a/\pi}{(a+h)}} = e^{-\lim_{h\to 0}\frac{2a}{\pi}\times\frac{1}{a+h}} = e^{-2/\pi}$$

Sol.18 D

$$\lim_{n\to\infty}\frac{1}{n^4}\left([1^3x]+[2^3x]+.....+[n^3x]\right)$$

$1^3x - 1 < [1^3x] \le 1^3x$

$2^3x - 1 < [2^3x] \le 2^3x$

$3^3x - 1 < [3^3x] \le 3^3x$

$\vdots \quad\quad \vdots \quad\quad \vdots$

$n^3x - 1 < [n^3x] \le n^3x$ so,

$(1^3x + ...+ n^3x) - n < [1^3x] ...+ [n^3x] \le 1^3x + 2^3x + ... n^3x$

$$\lim_{n\to\infty}\frac{(1^3x+...+n^3x)-n}{n^4} < \lim_{n\to\infty}\ell \le \lim_{n\to\infty}\frac{1^3x+...+n^3x}{n^4}$$

$$\lim_{n\to\infty}\frac{\left(\frac{n(n+1)}{2}\right)^2x-n}{n^4} < \lim_{n\to\infty}\ell \le \lim_{n\to\infty}\frac{\left(\frac{n(n+1)}{2}\right)^2x}{n^4}$$

$$\lim_{n\to\infty}\left(\frac{n^4\left(1+\frac{1}{n}\right)^2x}{4n^4}-\frac{1}{n^3}\right) < \lim_{n\to\infty}\ell \le \lim_{n\to\infty}\frac{n^4\left(1+\frac{1}{n}\right)^2x}{4n^4}$$

$$\frac{x}{4}-0 < \lim_{n\to\infty}\ell < \frac{x}{4}$$ So, $$\lim_{n\to\infty}\ell = \frac{x}{4}$$

Sol.19 D

$$\lim_{n\to\infty}\frac{5^{n+1}+3^n-2^{2n}}{5^n+2^n+3^{2n+3}}$$

$$= \lim_{n\to\infty}\frac{5^n\left(5+(3/5)^n-(4/5)^n\right)}{9^n\left((5/9)^n+(2/9)^n+27\right)}$$

$$= \lim_{n\to\infty}\frac{5}{27}\times\left(\frac{5}{9}\right)^n = 0$$

Sol.20 B

Given : $f(x) = \begin{cases} x-1, & x\ge 1 \\ 2x^2-2, & x<1\end{cases}$

$g(x) = \begin{cases} x+1, & x>0 \\ -x^2+1, & x\le 0\end{cases}$ & $h(x) = |x|$

LHL : $\lim_{x\to 0^-} f(g(h(0^-))) = \lim_{x\to 0^-} f(g(0^+))$

$= \lim_{x\to 0^-} f(1^+) = 0$

RHL : $\lim_{x\to 0^+} f(g(h(x))) = \lim_{x\to 0^+} f(g(h(0^+)))$

$= \lim_{x\to 0^+} f\{g(0^+)\} = 0$

Sol.21 C

$$\lim_{x\to\frac{\pi}{2}}\frac{\left(1-\tan\frac{x}{2}\right)(1-\sin x)}{\left(1+\tan\frac{x}{2}\right)(\pi-2x)^3}$$ Put $x = \left(\frac{\pi}{2}+h\right)$

$$= \lim_{h\to 0}\frac{1-\tan\left(\frac{\pi}{4}+\frac{h}{2}\right)}{1+\tan\left(\frac{\pi}{4}+\frac{h}{2}\right)}\times\frac{1-\sin\left(\frac{\pi}{2}+h\right)}{\left(\pi-\frac{2\pi}{2}-2h\right)^3}$$

$$= \lim_{h\to 0}\frac{1-\left\{\frac{1+\tan\frac{h}{2}}{1-\tan\frac{h}{2}}\right\}}{1+\left\{\frac{1+\tan\frac{h}{2}}{1-\tan\frac{h}{2}}\right\}}\left(\frac{1-\cos h}{-8h^3}\right)$$

$$= \lim_{h\to 0}\left(\frac{-2\tan(h/2)}{-2\times 8h^3}\right)\times(2\sin^2(h/2))$$

$$= \frac{1}{4}\lim_{h\to 0}\left(\frac{\tan h/2}{(h/2).2}\right)\times\left(\frac{\sin^2(h/2)}{h^2/4\times 4}\right) = \frac{1}{32}$$

Sol.22 C

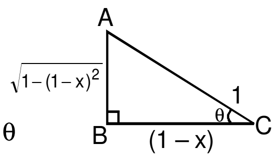
$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right)+\sin\left(\frac{1}{x^2}\right), & x\neq 0 \\ 0, & x = 0 \end{cases}$$

$$\Rightarrow \lim_{x\to\infty} f(x) = \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x^2}\right)$$

Put $x = \frac{1}{y} \Rightarrow \lim_{y\to 0}\left(\frac{\sin y}{y}\right) + \lim_{y\to 0}\sin y^2 = 1+0 = 1$

Sol.23 B

$$\lim_{x\to 0^+}\frac{\cos^{-1}(1-x)}{\sqrt{x}}$$



Let $\cos^{-1}(1-x) = \theta$

$(1-x) = \cos\theta$

$\sin\theta = \sqrt{1-\cos^2\theta} \Rightarrow \sin\theta = \sqrt{1-(1-x)^2}$

$\Rightarrow \theta = \sin^{-1}\sqrt{1-(1-x)^2} = \sin^{-1}\sqrt{2x-x^2}$

so, $\lim_{x\to 0^+}\frac{\cos^{-1}(1-x)}{\sqrt{x}} = \lim_{x\to 0^+}\frac{\sin^{-1}\sqrt{2x-x^2}}{\sqrt{x}}$

$$= \lim_{h\to 0}\frac{\sin^{-1}\sqrt{2h-h^2}}{\sqrt{2h-h^2}}\times\frac{\sqrt{2h-h^2}}{\sqrt{h}} = \sqrt{2}$$

Sol.24 C

Given α & β are the roots of $ax^2 + bx + c = 0$

$\lim_{x\to\alpha}(1+ax^2+bx+c)^{\frac{1}{(x-\alpha)}}$ It is of the form 1^∞,

$$\ell = e^{\lim_{x\to\alpha}\frac{1}{(x-\alpha)}(ax^2+bx+c)} = e^{\lim_{x\to\alpha}\frac{a}{x-\alpha}\times\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)}$$

$$= e^{\lim_{x\to\alpha}\frac{a(x-\alpha)(x-\beta)}{(x-\alpha)}} = e^{a(\alpha-\beta)}$$

Sol.25 A

$$\lim_{x\to\frac{\pi}{2}}\frac{1-2^{\cos x}}{x\left(x-\frac{\pi}{2}\right)}\,2^{\cos x}$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{2^{\sin\left(\frac{\pi}{2}-x\right)}-1}{\left(\frac{\pi}{2}-x\right)x.2^{\cos x}} = \frac{2\ln 2}{\pi}$$

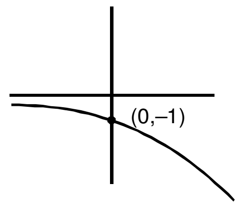
Sol.26 B

$\lim_{x\to 0}(\cos mx)^{\frac{n}{x^2}}$ since it is of the form 1^∞, so,

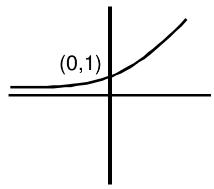
$$\ell = e^{\lim_{x\to 0}\frac{n}{x^2}(\cos mx-1)} = e^{\lim_{x\to 0}\frac{-2n\sin^2\left(\frac{mx}{2}\right)}{x^2\times\frac{m^2}{4}\times\frac{4}{m^2}}} = e^{\left(-\frac{m^2n}{2}\right)}$$

Sol.27 A

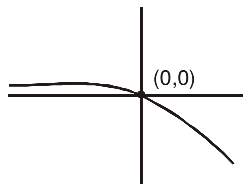
$$\lim_{x\to 0}\left[(1-e^x)\frac{\sin x}{|x|}\right]$$



$$\lim_{x\to 0^+}\left[(1-e^x)\frac{\sin x}{x}\right]$$



$1 - [-0.0999] = -1$



Sol.28 B

$$\lim_{x\to 0}\frac{(4^x-1)^3}{\sin\left(\frac{x}{p}\right)\ell n\left(1+\frac{x^2}{3}\right)}$$

$$= \lim_{x\to 0}\frac{(4^x-1)^3}{x^3}\times\frac{x^3}{\sin\left(\frac{x}{p}\right)}\times\frac{1}{\ell n\left(1+\frac{x^2}{3}\right)}$$

$$= \lim_{x\to 0}\left(\frac{4^x-1}{x}\right)^3\times\frac{1}{\frac{\sin(x/p)}{(x/p)\times p}}\times\frac{1}{\ell n\frac{(1+x^2/3)}{(x^2/3)\times 3}}$$

$= (\ell og\ 4)^3 \times p \times 3 \Rightarrow 3p(\log 4)^3$

Sol.29 C

$$f(x) = \begin{cases} \dfrac{\tan^2[x]}{(x^2 - [x]^2)} & ; \ x > 0 \\ 1 & ; \ x = 0 \\ \sqrt{\{x\} \cot \{x\}} & ; \ x < 0 \end{cases}$$

$$\text{RHL : } \lim_{x \to 0^-} \sqrt{\{x\} \cot \{x\}} = \sqrt{1 \times \cot 1} = \sqrt{\cot 1}$$

$$\text{LHL : } \lim_{x \to 0^+} \frac{\tan^2[x]}{(x^2 - [x]^2)} = 0 \ \Rightarrow \cot^{-1} \left(\lim_{x \to 0^-} f(x) \right)^2 = 1$$

Sol.30 C

$$\ell = \lim_{x \to 0} \frac{1 - \cos x \sqrt{\cos 2x}}{x^2}$$

$$\ell = \lim_{x \to 0} \frac{(1 - \cos x \sqrt{\cos 2x})\,(1 + \cos x \sqrt{\cos 2x})}{x^2\,(1 + \cos x \sqrt{\cos 2x})}$$

$$\ell = \lim_{x \to 0} \frac{1 - \cos^2 x . \cos 2x}{x^2\,(1 + 1\sqrt{1})}$$

$$\ell = \lim_{x \to 0} \frac{1 - \cos^2 x (1 - 2\sin^2 x)}{2x^2}$$

$$\ell = \lim_{x \to 0} \frac{1 - \cos^2 x + 2\sin^2 x \cos^2 x}{2x^2}$$

$$\ell = \lim_{x \to 0} \frac{\sin^2 x}{2x^2} + \lim_{x \to 0} \frac{\sin^2 x}{x^2} . \cos^2 x$$

$$\ell = \frac{1}{2} + 1 = \frac{3}{2}$$

Sol.31 A

$$\lim_{x \to \infty} x - x^2 \,\ell n \left(1 + \frac{1}{x}\right)$$

Put $x = \dfrac{1}{y}$, so $(x \to \infty)$

will be replaced by $(y \to 0)$

$$\Rightarrow \lim_{y \to 0} \frac{1}{y} - \frac{1}{y^2} \,\ell n\,(1 + y) = \lim_{y \to 0} \frac{y - \ell n\,(1 + y)}{y^2}$$

Applying L-Hospital rule : $\displaystyle \lim_{y \to 0} \frac{1 - \dfrac{1}{1 + y}}{2y}$

Again applying L – Hospital rule : $\displaystyle \lim_{y \to 0} \frac{\dfrac{1}{(1 + y)^2}}{2} = \frac{1}{2}$

Sol.32 D

$$\lim_{n \to \infty} \left(\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2 + 1}} + \ldots\ldots + \frac{1}{\sqrt{n^2 + 2n}} \right)$$

$$\Rightarrow \lim_{n \to \infty} \frac{(2n + 1)}{\sqrt{n^2 + 2n}} \le \ell \le \frac{(2n + 1)}{\sqrt{n^2}}$$

$$\Rightarrow \quad \lim_{n \to \infty} \frac{(2n + 1)}{n\sqrt{1 + \dfrac{2n}{n^2}}} \le \ell \le \lim_{n \to \infty} \left(\frac{2n + 1}{n} \right)$$

$$\Rightarrow \quad 2 \le \ell \le 2 \qquad \Rightarrow \qquad \ell = 2$$

Sol.33 C

$$f(x) = \lim_{t \to 0} \left(\frac{2x}{\pi} \cot^{-1} \left(\frac{x}{t^2} \right) \right)$$

<u>**Case –1**</u> : $\to$ If $x < 0$,
Let $x = -k$, where $k > 0$.

$$\text{so, } \frac{2x}{\pi} \cot^{-1} \left(\frac{x}{t^2} \right) = \frac{-2k}{\pi} \cot^{-1} \left(\frac{-k}{t^2} \right)$$

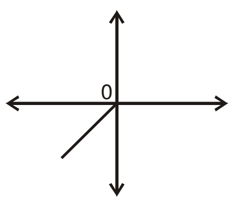
$$= \frac{-2k}{\pi} \left(\frac{\pi}{2} + \tan^{-1} \frac{k}{t^2} \right)$$

$$\lim_{t \to 0} \frac{2x}{\pi} \cot^{-1} \left(\frac{x}{t^2} \right) = \lim_{t \to 0} - \frac{2k}{\pi} \left(\frac{\pi}{2} + \tan^{-1} \frac{k}{t^2} \right)$$

$$= -k - k \qquad \Rightarrow \qquad 2x$$

<u>**Case – 2**</u> : $\to$ If $x > 0$,

$$\lim_{t \to \infty} \frac{2x}{\pi} \cot^{-1} \left(\frac{x}{t^2} \right) = 0$$



i.e. x-axis.
so, graph of f(x) will be

Sol.34 A

Given that , $\displaystyle \lim_{x \to \infty} f(x)$ is finite & non zero.

$$\text{Also, } \lim_{x \to \infty} \left(f(x) + \frac{3f(x) - 1}{f^2(x)} \right) = 3$$

Let $\lim_{x\to\infty} f(x) = \ell$ so $\lim_{x\to\infty}\left(f(x)+\frac{3f(x)-1}{f^2(x)}\right) = 3$

$\Rightarrow \ell + \frac{3\ell-1}{\ell^2} - 3 = 0 \Rightarrow (\ell-1)^3 = 0 \quad \Rightarrow \ell = 1$

Sol.35 B

$$\lim_{x\to 0}\frac{\cos(\sin x)-\cos x}{x^4}$$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{\sin x + x}{2}\right)\sin\left(\frac{x-\sin x}{2}\right)}{x^4}$$

$$= \lim_{x\to 0}\frac{2\sin\left\{\frac{\sin x+x}{2}\right\}}{\left\{\frac{\sin x+x}{2}\right\}} \times \frac{\left(\frac{\sin x+x}{2}\right)}{x^4} \times \frac{\sin\left(\frac{x-\sin x}{2}\right)}{\left(\frac{x-\sin x}{2}\right)}\left(\frac{x-\sin x}{2}\right)$$

$$= \lim_{x\to 0} 2\left(\frac{x+\sin x}{2}\right) \times \left(\frac{x-\sin x}{2}\right) \times \frac{1}{x^4}$$

$$= \lim_{x\to 0}\frac{1}{2}\left(\frac{x+\sin x}{x}\right)\left(\frac{x-\sin x}{x^3}\right)$$

Let $\ell = \lim_{x\to 0}\frac{x-\sin x}{x^3}$

$$= \lim_{x\to 0}\left(\frac{1-\cos x}{3x^2}\right) = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{3\frac{x^2}{4}\times 4} = \frac{1}{6}$$

Now, $\lim_{x\to 0}\frac{1}{2}\left(\frac{x+\sin x}{x}\right) \times \frac{1}{6}$

$$\frac{1}{12}\lim_{x\to 0}\left(\frac{x}{x}+\frac{\sin x}{x}\right) = \frac{1}{12}(1+1) = \frac{1}{6}$$

Sol.36 B

$$\lim_{x\to\infty}\frac{e^x\left[\left(2^{x^n}\right)^{\frac{1}{e^x}}-\left(3^{x^n}\right)^{\frac{1}{e^x}}\right]}{x^n}$$

$$= \lim_{x\to\infty}\frac{(2^{x^n/e^x}-1)-(3^{x^n/e^x}-1)}{(x^n/e^x)}$$

Now, $\lim_{x\to\infty}\frac{x^n}{e^x} = 0$

So, $\lim_{x\to\infty}\left\{\frac{2^{(x^n/e^x)}-1}{(x^n/e^x)}\right\} - \lim_{x\to\infty}\left\{\frac{3^{(x^n/e^x)}-1}{(x^n/e^x)}\right\}$

$= \log_e 2 - \log_e 3 = \log_e\left(\frac{2}{3}\right)$

Sol.37 B

$\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x} = e^2$ It is of the form 1^∞

$$\ell = e^{\lim_{x\to\infty} 2x\left(1+\frac{a}{x}+\frac{b}{x^2}-1\right)} = e^{\lim_{x\to\infty}\left(2a+\frac{2b}{x}\right)} = e^{2a}$$

but $\ell = e^2$, so $e^{2a} = e^2 \Rightarrow a = 1$ & $b \in R$

Sol.38 A

$$\lim_{x\to\infty}\frac{\log_x n-[x]}{[x]} = \lim_{x\to\infty}\frac{\log n-[x]\log x}{[x]\log x}$$

$$\lim_{x\to\infty}\frac{\log n}{[x]\log x} - 1 = -1$$

Sol.39 A

$\ell = \lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+3}\right)^x$ It is of form 1^∞

$$\ell = e^{\lim_{x\to\infty} x\left\{\frac{x^2+5x+3}{x^2+x+3}-1\right\}} = e^{\lim_{x\to\infty} x\left(\frac{4x}{x^2+x+3}\right)}$$

$$= e^{\lim_{x\to\infty}\left(\frac{4}{1+1/x+3/x^2}\right)} = e^4$$

Sol.40 D

Given that α & β are the roots of $ax^2 + bx + c = 0$

$$\ell = \lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$$

$$\ell = \lim_{x\to\alpha}\frac{2\sin^2\left(\frac{ax^2+bx+c}{2}\right)}{(x-\alpha)^2}$$

$$\ell = \lim_{x\to\alpha} \frac{2\sin^2\left(\frac{a(x-\alpha)(x-\beta)}{2}\right)}{\frac{(x-\alpha)^2}{4}\times 4\times\frac{(x-\beta)^2}{(x-\beta)^2}\times\frac{a^2}{a^2}}$$

$$\ell = \frac{2}{4}\times a^2 \lim_{x\to\alpha}(x-\beta)^2 = \frac{a^2}{2}(\alpha-\beta)^2$$

Sol.41 C

$$\lim_{x\to 0}\frac{2\left\{\sqrt{3}\sin\left(\frac{\pi}{6}+x\right)-\cos\left(\frac{\pi}{6}+x\right)\right\}}{x\sqrt{3}\,(\sqrt{3}\cos x-\sin x)}$$

$$= \lim_{x\to 0}\frac{2\times 2\left\{\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{6}+x\right)-\frac{1}{2}\cos\left(\frac{\pi}{6}+x\right)\right\}}{x\sqrt{3}\,(\sqrt{3}\cos x-\sin x)}$$

$$= \lim_{x\to 0}\frac{4\times\sin\left(\frac{\pi}{6}+x-\frac{\pi}{6}\right)}{x\sqrt{3}\,(\sqrt{3}\cos x-\sin x)}$$

$$= \lim_{x\to 0} 2\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos x-\frac{1}{2}\sin x\right)}$$

$$= 2\lim_{x\to 0}\frac{1}{\sqrt{3}\cos\left(\frac{\pi}{6}+x\right)} = \frac{4}{3}$$

Sol.42 A

$$\lim_{n\to\infty}\frac{1.n+2(n-1)+......+n.1}{1^2+2^2+.......+n^2}$$

consider the numerator, it can be written as follows

$$\sum_{r=1}^{n} r\,(n-r+1) = (n+1)\sum_{r=1}^{n} r-\sum_{r=1}^{n} r^2$$

$$= (n+1)\,(1+2+3+....+n)-(1^2+2^2+...+n^2)$$

$$= \frac{(n+1).n(n+1)}{2}-\frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(n+2)}{6}$$

$$\text{Now}\lim_{n\to\infty}\frac{\frac{n(n+1)(n+2)}{6}}{\frac{n(n+1)(2n+1)}{6}} = \lim_{n\to\infty}\frac{(n+2)}{(2n+1)} = \frac{1}{2}$$

Sol.43 B

$(\tan\alpha)\,x + (\sin\alpha)y = \alpha$

$[(\alpha\,\text{cosec}\,\alpha)x + (\cos\alpha)y = 1]\times\tan\alpha$

$$x = \frac{\alpha-\tan\alpha}{\tan\alpha-\alpha\sec\alpha}$$

$$x = \lim_{\alpha\to 0}\frac{\alpha-\tan\alpha}{\frac{1}{\cos\alpha}(\sin\alpha-\alpha)}\times\frac{1/\alpha^3}{1/\alpha^3}$$

$$x = \lim_{\alpha\to 0}\frac{-1/3}{-1/6}\times\cos\alpha = 2$$

and $$y = \lim_{\alpha\to 0}\frac{\tan^2\alpha-\alpha^2\sec\alpha}{\sin\alpha(\tan\alpha-\alpha\sec\alpha)}$$

$$= \lim_{\alpha\to 0}\frac{\tan^2\alpha-\alpha^2\sec\alpha+\alpha^2-\alpha^2}{\frac{\sin\alpha}{\alpha}\,\frac{(\tan\alpha-\alpha\sec\alpha)}{\alpha^3}.\alpha^4}$$

$$= \lim_{\alpha\to 0}\frac{\left[\frac{\tan\alpha}{\alpha}+1\right]\left[\frac{\tan\alpha-\alpha}{\alpha^3}\right]-\frac{1-\cos\alpha}{\cos\alpha.\alpha^2}}{\left(\frac{\sin\alpha}{\alpha}\right)\left(\frac{\sin\alpha-\alpha}{\alpha^3}\right)\frac{1}{\cos\alpha}} = -1$$

Sol.44 C

$$\lim_{x\to 0}\left\{\cot\left(\frac{\pi}{4}+x\right)\right\}^{(\text{cosec}\,x)}$$

$$\ell = e^{\lim_{x\to 0}\text{cosec}\,x\left(\cot\left(\frac{\pi}{4}+x\right)-1\right)}$$

$$= e^{\lim_{x\to 0}\text{cosec}\,x\left(\frac{1}{\tan(\frac{\pi}{4}+x)}-1\right)}$$

$$= e^{\lim_{x\to 0}\text{cosec}\,x\left(\frac{1-\tan x-1-\tan x}{1+\tan x}\right)}$$

$$= e^{\lim_{x\to 0}-\text{cosec}\,x\left(\frac{2\tan x}{1+\tan x}\right)} = e^{-\lim_{x\to 0}\frac{2}{\cos x+\sin x}} = e^{-2}$$

Sol.45 C

$$\ell = \lim_{x\to 0}\frac{\sin\cdot[\cos x]}{1+[\cos x]}$$

$$\text{LHL : }\lim_{h\to 0}\frac{\sin[\cos(0-h)]}{1+[\cos(0-h)]} \Rightarrow \lim_{h\to 0}\frac{\sin 0}{1+0} = 0$$

$(\because [\cos(0-h)] = 0)$

RHL : $\lim\limits_{h\to 0} \dfrac{\sin[\cos(0+h)]}{1+[\cos(0+h)]} \Rightarrow \lim\limits_{h\to 0} \dfrac{\sin 0}{1+0} = 0$

so, $\lim\limits_{x\to 0} \dfrac{\sin[\cos x]}{1+[\cos x]} = 0$

Sol.46 A

1^∞ From

$$\ell = \lim_{x\to\infty} x\left[\sin\frac{1}{x}+\cos\frac{1}{x}-1\right]$$

put x = 1/t

$$= \lim_{t\to 0} \frac{1}{t}\left[\sin t+\cos t-1\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[2\sin\frac{t}{2}\cos\frac{t}{2}-2\sin^2\frac{t}{2}\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\; 2\sin\frac{t}{2}\left[\cos\frac{t}{2}-\sin\frac{1}{2}\right]$$

$= 1$

$= e^1$

Sol.47 C

$$\ell = \lim_{x\to 0} \frac{e^{-nx}+e^{nx}-2\cos\frac{nx}{2}-kx^2}{(\sin x-\tan x)}$$

$$= \lim_{x\to 0} \frac{1+\frac{nx}{1!}+\frac{n^2x^2}{2!}+\frac{n^3x^3}{3!}+1-\frac{nx}{1!}+\frac{n^2x^2}{2!}-\frac{n^3x^3}{3!}-2\cos\frac{nx}{2}-kx^2}{\left\{\frac{(\sin x-\tan x)}{x^3}\right\}x^3}$$

$$= \lim_{x\to 0} \frac{2+n^2x^2+\frac{n^4x^4}{12}-2\left(1-\frac{n^2x^2}{4.2!}+\frac{n^4x^4}{16.4!}\right)-kx^2}{x^3\left(\frac{\sin x-\tan x}{x^3}\right)}$$

$$= \frac{\left(\frac{5n^2}{4}-k\right)x^2+Mx^4}{x^3\left(\frac{\sin x-\tan x}{x^3}\right)}$$

For existence of limit, $\frac{5n^2}{4} = k$, now check value of n & k from options.

Sol.48 D

$\lim\limits_{x\to a_m} (A_1 . A_2 \,..........\, A_n),\ 1 \le m \le n$

$$\lim_{x\to a_m} \frac{x-a_1}{|x-a_1|}.\frac{x-a_2}{|x-a_2|}\cdots\frac{x-a_m}{|x-a_m|}\cdots\frac{x-a_n}{|x-a_n|}$$

$$\lim_{x\to a_m} (1)^{m-1}\times\frac{x-a_m}{|x-a_m|}\times(-1)^{n-m}$$

RHL: $(-1)^{n-m}\times 1$

LHL: $(-1)^{n-m}\times(-1)$

so, limit does not exists.

Sol.49 A

$$f(x) = \frac{\ell n(x^2+e^x)}{\ell n(x^4+e^{2x})}$$

$$\ell = \lim_{x\to\infty} \frac{\ell n(x^2+e^x)}{\ell n(x^4+e^{2x})}$$

$$= \lim_{x\to\infty} \frac{\ell n e^x+\ell n\left(1+\frac{x^2}{e^x}\right)}{\ell n e^{2x}+\ell n\left(1+\frac{x^4}{e^{2x}}\right)}$$

$$= \lim_{x\to\infty} \frac{x+\ell n\left(1+\frac{x^2}{e^x}\right)}{2x+\ell n\left(1+\frac{x^4}{e^{2x}}\right)} = \frac{1}{2}$$

$$m = \lim_{x\to-\infty} \frac{\ell n(x^2+e^x)}{\ell n(x^4+e^{2x})} \qquad \left(\lim_{x\to\infty}\frac{x^2}{e^x}\to\infty\right)$$

so, we will be create $\dfrac{e^x}{x^2}$ now,

$$m = \lim_{x\to-\infty} \frac{\ell n\, x^2+\ell n\,(1+e^x/x^2)}{\ell n\, x^4+\ell n\,(1+e^{2x}/x^4)} = \frac{1}{2}$$

hence, $\ell = m$

Sol.50 C

Given a = min $\{x^2 + 2x + 3,\ x \in R\}$

& b = $\lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\theta^2}$

Let $f(x) = x^2 + 2x + 3$. Its discriminant is less than zero. Its min. value is $\dfrac{-D}{4a} = \dfrac{-(-8)}{4\times 1} = 2$

$$\text{so } a = 2 \ \& \ b = \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2} = \lim_{\theta\to 0}\frac{2\sin^2\left(\frac{\theta}{2}\right)}{\frac{\theta^2}{4}.4} = \frac{1}{2}$$

$$\text{Now } \sum_{r=0}^{n} a^r\, b^{n-r} = \sum_{r=0}^{n} 2^r \left(\frac{1}{2}\right)^{n-r}$$

$$= \frac{1}{2^n}\sum_{r=0}^{n} 2^{2r} = \frac{1}{2^n}\left(2^0 + 2^2 + \ldots\ldots.. + 2^{2n}\right)$$

$$= \frac{1}{2^n}\,\frac{(2^2)^{n+1}-1}{(2^2-1)} = \frac{4^{n+1}-1}{3\cdot 2^n}.$$

Sol.51 B

$$\lim_{x\to\infty}\left(\frac{\cos h\frac{\pi}{x}}{\cos\frac{\pi}{x}}\right)^{x^2}, \text{ given, } \frac{e^t+e^{-t}}{2} = \cos ht$$

Put $x = \frac{1}{y} \Rightarrow \lim_{y\to 0}\left(\frac{\cosh\ \pi y}{\cos\ \pi y}\right)^{\frac{1}{y^2}}$. It is 1^∞ form,

$$\ell = e^{\lim_{y\to 0}\frac{1}{y^2}\left(\frac{e^{\pi y}+e^{-\pi y}-2\cos\pi y}{2\cos\pi y}\right)}$$

$$= e^{\lim_{y\to 0}\frac{1}{y^2}\left(\frac{e^{\pi y}+e^{-\pi y}-2+2-2\cos\pi y}{2\cos\pi y}\right)}$$

$$= e^{\lim_{y\to 0}\left[\frac{1}{y^2}\left(\frac{e^{\pi y}+e^{-\pi y}-2}{2\cos\pi y}\right)+\frac{2(1-\cos\pi y)}{2y^2\cos\pi y}\right]}$$

$$= e^{\lim_{y\to 0}\frac{e^{\pi y}+e^{-\pi y}-2}{2y^2\cos\pi y}+\lim_{y\to 0}\frac{2\cdot 2\sin^2\left(\frac{\pi y}{2}\right)}{2y^2(\cos\pi y)}}$$

$$= e^{\lim_{y\to 0}\left[\frac{e^{\pi y}+e^{-\pi y}-2}{2y^2\cos\pi y}\right]+\left[\frac{\pi^2}{2}\right]}$$

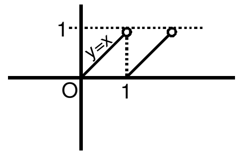
using L – Hospital rule twice we get $\ell = e^{\pi^2}$

Sol.52 A

$$\lim_{x\to 0^+}\frac{\left(\frac{\pi}{2}-\cot^{-1}\{x\}\right)x}{\mathrm{sgn}(x)-\cos x} = \lim_{x\to 0^+}\frac{x\tan^{-1}\{x\}}{\frac{|x|}{x}-\cos x}$$

Now for $x \to 0^+$, $|x| = x$ & $\{x\} = x$

$$\text{so, } \lim_{x\to 0^+}\frac{x\tan^{-1}x}{(1-\cos x)}$$



$$= \lim_{x\to 0^+}\frac{x\tan^{-1}x}{2\sin^2\left(\frac{x}{2}\right)} = \lim_{x\to 0^+}\frac{x\tan^{-1}x}{2.\frac{\sin^2\left(\frac{x}{2}\right)}{\left(x^2/4\right)}\times\frac{x^2}{4}} = 2$$

Sol.53 D

$$\lim_{n\to\infty}\left[(1+x)(1+x^2).......(1+x^{2^n})\right]$$

$$= \lim_{n\to\infty}\left[\frac{(1-x)(1+x)........(1+x^{2^n})}{(1-x)}\right] = \lim_{n\to\infty}\left(\frac{1-x^{2^{n+1}}}{1-x}\right)$$

$$= \frac{1}{(1-x)}\lim_{n\to\infty}\left(1-x^{2^{n+1}}\right) = \frac{1}{(1-x)}$$

Sol.54 A

$$\lim_{x\to 0}\frac{x^3}{\sqrt{a+x}\,(bx-\sin x)} = 1$$

$$= \lim_{x\to 0}\frac{x^3}{\sqrt{a+x}\left(bx-\left(x-\frac{x^3}{3!}+....\right)\right)} = 1$$

$$= \frac{1}{\sqrt{a}}\lim_{x\to 0}\frac{x^3}{x(b-1)+\frac{x^3}{3!}+....} = 1$$

If limit exists then, $b - 1 = 0 \Rightarrow b = 1$

$$\text{so, } \frac{1}{\sqrt{a}}\lim_{x\to 0}\frac{x^3}{\frac{x^3}{3!}+....} = 1$$

$$\Rightarrow \frac{6}{\sqrt{a}} = 1 \Rightarrow a = 36 \quad \text{so, } \left.\begin{matrix} a = 36 \\ b = 1 \end{matrix}\right\}$$

Sol.55 A

$$\lim_{n\to\infty}\frac{1^2 n + 2^2(n-1) + \ldots\ldots\ldots. + n^2.1}{1^3+2^3+\ldots\ldots+n^3}$$

$$= \lim_{n\to\infty} \frac{\sum_{r=1}^{n} r^2(n-r+1)}{1^3+2^3+......+n^3}$$

$$= \lim_{n\to\infty} \frac{n\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r^2}{\left(\frac{n(n+1)}{2}\right)^2}$$

$$= \lim_{n\to\infty} \frac{\frac{(n+1)n\,(n+1)(2n+1)}{6} - \left(\frac{n(n+1)}{2}\right)^2}{\left(\frac{n(n+1)}{2}\right)^2}$$

$$= \lim_{n\to\infty} \frac{\frac{n(n+1)^2(2n+1)}{6} - \left(\frac{n(n+1)}{2}\right)^2}{\left(\frac{n(n+1)}{2}\right)^2}$$

$$= \lim_{n\to\infty} \frac{n(n+1)^2}{24}\ [8n + 4 - 6n] \times \frac{4}{n^2(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{2n+4}{6n} \Rightarrow \lim_{n\to\infty} \frac{2+4/n}{6} = \frac{1}{3}$$

Sol.56 B

$$\lim_{x\to 0} \frac{\sin(6x^2)}{\ell n\cos(2x^2-x)}$$

$$= \lim_{x\to 0} \frac{\sin(6x^2)}{(6x^2)}.(6x^2) \times \frac{1}{\ell n\left(1-2\sin^2\left(\frac{2x^2-x}{2}\right)\right)}$$

$$= \lim_{x\to 0} \frac{6x^2}{\frac{\ell n\left(1-2\sin^2\left(\frac{2x^2-x}{2}\right)\right)}{-2\sin^2\left(\frac{2x^2-x}{2}\right)} \times -2\sin^2\left(\frac{2x^2-x}{2}\right)}$$

$$= -3 \lim_{x\to 0} \left(\frac{2x}{2x^2-x}\right)^2 \times \frac{\left(\frac{2x^2-x}{2}\right)^2}{\sin^2\left(\frac{2x^2-x}{2}\right)} = -12$$

Sol.57 A

$$\ell = \lim_{n\to\infty} \left[\frac{n}{n^2} + \frac{n}{n^2+1} + ... + \frac{n}{2n^2-2n+1}\right]$$

$$\ell = \lim_{n\to\infty} \sum_{r=0}^{n-1} \left(\frac{n}{n^2+r^2}\right) = \lim_{n\to\infty} \sum_{r=0}^{n-1} \left(\frac{1}{1+(r/n)^2}\right)\frac{1}{n}$$

Let $\frac{r}{n} = x \Rightarrow \frac{1}{x} = dx$

at r = 0, $x = \lim_{n\to\infty} \frac{0}{n} = 0$

at r = n – 1, $x = \lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} \left(1-\frac{1}{n}\right) = 1$

Now, $\ell = \int_0^1 \frac{dx}{1+x^2} = (\tan^{-1} x)_0^1 = \frac{\pi}{4}$

Sol.58 C

$$\lim_{x\to 0} \frac{e^{-x^2/2} - \cos x}{x^3 \sin x}$$

$$= \lim_{x\to 0} \frac{1+\frac{1}{2}\left(-\frac{x^2}{2}\right)^2 + \left(-\frac{x^2}{2}\right) + ... - \left(1-\frac{x^2}{21}+\frac{x^4}{31}\right)}{x^4.\left(\frac{\sin x}{x}\right)}$$

$$= \lim_{x\to 0} \frac{x^4\left(\frac{3-1}{24}\right) + ...}{x^4} = \frac{1}{12}$$

EXERCISE – II HINT & SOLUTION

Sol.1 A,B,C

given $f(x) = \dfrac{x^2-9x+20}{x-[x]}$

$\lim\limits_{x\to5^-} f(x) = \lim\limits_{x\to5^-} \dfrac{x^2-9x+20}{x-[x]}$

$= \lim\limits_{x\to5^-} \dfrac{x^2-9x+20}{x-4} = \lim\limits_{x\to5^-} \dfrac{(x-5)(x-4)}{(x-4)} = 0$

& $\lim\limits_{x\to5^+} f(x) = \lim\limits_{x\to5^+} \dfrac{x^2-9x+20}{x-[x]}$

$= \lim\limits_{x\to5^+} \dfrac{(x-4)(x-5)}{(x-5)} = 1$

since LHL $\neq$ RHL.

Sol.2 A,B

$\lim\limits_{x\to-1} f(x) = \lim\limits_{x\to-1} \dfrac{\cos 2-\cos 2x}{x^2-|x|}$

$= \lim\limits_{x\to-1} \dfrac{\cos 2-\cos 2x}{x^2+x} \quad \left(\dfrac{0}{0}\right)$

$= \lim\limits_{x\to-1} \dfrac{\cos 2-1+2\sin^2 x}{x(x+1)}$

$= \lim\limits_{x\to-1} \dfrac{2\sin^2 x-2\sin^2 1}{x(x+1)}$

$= 2\lim\limits_{x\to-1} \dfrac{\sin(x+1)\sin(x-1)}{x(x+1)} = -2\sin(-2)$

$= 2\sin 2$

& $\lim\limits_{x\to1} \dfrac{\cos 2-\cos 2x}{x^2-x} \quad \left(\dfrac{0}{0}\right)$

$= \lim\limits_{x\to1} \dfrac{\cos 2-1+2\sin^2 x}{x(x-1)}$

$= \lim\limits_{x\to1} \dfrac{2\sin(x+1)\sin(x-1)}{x(x-1)} = 2\sin 2$

Sol.3 A,B

$f(x) = \dfrac{\sqrt{x^2+2}}{3x-6} \Rightarrow \lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+2}}{3x-6}$

$= \lim\limits_{x\to-\infty} \dfrac{|x|\sqrt{1+\frac{2}{x^2}}}{3x-6} = \lim\limits_{x\to-\infty} \dfrac{-x\sqrt{1+\frac{2}{x^2}}}{x(3-6/x)} = -\dfrac{1}{3}$

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+2}}{3x-6} = \lim\limits_{x\to\infty} \dfrac{x\sqrt{1+\frac{2}{x^2}}}{x\left(3-\frac{6}{x}\right)} = \dfrac{1}{3}$

Sol.4 A,B,C,D

$\lim\limits_{x\to0} (\cos x + a\sin bx)^{1/x} = e^2$

since it is 1^∞ form so result will be e^ℓ.

$\ell = \lim\limits_{x\to0} \dfrac{1}{x}(\cos x + a\sin bx - 1)$

$\ell = \lim\limits_{x\to0} \left[\dfrac{a\sin bx\times b}{x\times b} - \dfrac{x\times(1-\cos x)}{x\times x}\right]$

$\ell = ab \Rightarrow e^{ab} = e^2$

Sol.5 A,D

$\ell = \lim\limits_{x\to0} \dfrac{x(1+a\cos x)-b\sin x}{x^3}$

$\ell = \lim\limits_{x\to0} \dfrac{1+a\cos x}{x^2} - \lim\limits_{x\to0} \dfrac{b\sin x}{x^3}$

$\ell_1 = \lim\limits_{x\to0} \dfrac{1+a\cos x}{x^2} \quad \left(\dfrac{a+1}{0}\right)$

for existence of limit $a = -1$

$\ell_2 = \lim\limits_{x\to0} \dfrac{b\sin x}{x^3} \quad \left(\dfrac{0}{0}\right)$

so, b can be zero & for $a = -1$, $b = 0$

$\ell_1 = \dfrac{1}{2}$, $\ell_2 = 0$ so $\ell = 1/2$

Sol.6 A,D

$\lim\limits_{x\to0} \dfrac{\sin 2x + a\sin x}{x^3} = p$

$\lim\limits_{x\to0} \dfrac{2x-\frac{(2x)^3}{3!}+ax-\frac{ax^3}{3!}}{x^3} = p$

$$\lim_{x\to 0}\frac{(2+a)x-\dfrac{x^3}{3!}(a+2^3)}{x^3}=p$$

Now for existence of limit lower degree of x should not be present in N^r so to dissolve lower degrees a = –2

so for a = – 2, $\lim_{x\to 0}\dfrac{-\dfrac{6x^3}{3!}}{x^3}=p \Rightarrow p=-1$

Sol.7 B,C,D

$$\lim_{x\to 0}(1+ax+bx^2)^{\frac{2}{x}}=e^3$$

it is 1^∞ form

$$\ell=\lim_{x\to 0}\frac{2}{x}(1+ax+bx^2-1)=3$$

$$\ell=\lim_{x\to 0}\frac{2}{x}(a+bx)x=\lim_{x\to 0}(2a+2bx)$$

$\ell = 2a \quad\Rightarrow\quad a = 3/2$

Sol.8 A,B,C,D

$$\lim_{x\to\infty}\frac{(ax+1)^n}{x^n+A}\Rightarrow\lim_{x\to\infty}\frac{x^n\left(a+\dfrac{1}{x}\right)^n}{\left(1+\dfrac{A}{x^n}\right)x^n}=a^n,\ n\in N$$

$$\lim_{x\to\infty}\frac{(ax+1)^n}{x^n+A}\Rightarrow\lim_{x\to\infty}\frac{(ax+1)^0}{x^0+A}$$

Sol.9 A,C

$\lim_{x\to\infty}\left(1+\dfrac{1}{x^n}\right)^x$ is of 1^∞ for n > 1

so, $\ell=\lim_{x\to\infty}x\left(\dfrac{1}{x^n}\right)=\lim_{x\to\infty}\left(\dfrac{x}{x^n}\right)=0$

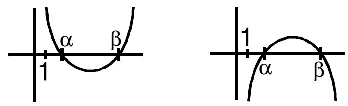
for 0 < n < 1 it is ∞

Sol.10 B,D

$ax^2+bx+c=0 <^{\alpha}_{\beta} \quad 1<\alpha<\beta$

$$\lim_{x\to x_0}\frac{|ax^2+bx+c|}{ax^2+bx+c}=1$$

B & D are incorrect



EXERCISE – III HINT & SOLUTION

Sol.1 $\lim_{x\to 1}\dfrac{\sqrt[13]{x}-\sqrt[7]{x}}{\sqrt[5]{x}-\sqrt[3]{x}} \Rightarrow \lim_{x\to 1}\dfrac{x^{\frac{1}{13}}-x^{\frac{1}{7}}}{x^{\frac{1}{5}}-x^{\frac{1}{3}}}\quad\left(\dfrac{0}{0}\right)$

Replace $(x\to 1+h)$

$$=\lim_{h\to 0}\frac{(1+h)^{\frac{1}{13}}-(1+h)^{\frac{1}{7}}}{(1+h)^{\frac{1}{5}}-(1+h)^{\frac{1}{3}}}$$

$$=\lim_{h\to 0}\frac{1+\dfrac{h}{13}-1-\dfrac{h}{7}}{1+\dfrac{h}{5}-1-\dfrac{h}{3}}=\frac{45}{91}$$

Sol.2 $\lim_{x\to 1}\dfrac{x^2-x\ell nx+\ell nx-1}{(x-1)}$

$$=\lim_{h\to 0}\frac{(1+h)^2-(1+h)\ell n(1+h)+\ell n(1+h)-1}{h}$$

$$=\lim_{h\to 0}\frac{h^2+2h-h\ell n(1+h)}{h}$$

$$=\lim_{h\to 0}\frac{h+2-\ell n(1+h)}{1}=2$$

Sol.3 $\lim_{x\to 1}\dfrac{\left[\sum_{k=1}^{100}x^k\right]-100}{(x-1)}$

$$=\lim_{x\to 1}\frac{x+x^2+x^3+\ldots+x^{100}-100}{(x-1)}$$

Now, put x = 1 + h

$$=\lim_{h\to 0}\frac{1+h+(1+h)^2+\ldots+(1+h)^{100}-100}{h}$$

$=\lim_{h\to 0}1+2+\ldots+100=5050$

($\because$ Applying binomial theorem)

Sol.4 $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{1-\tan x}{1-\sqrt{2}\sin x}$ Put $x = \dfrac{\pi}{4} + h$

$$= \lim_{h\to 0} \frac{1-\dfrac{1+\tan h}{1-\tan h}}{1-\sqrt{2}\left\{\sin\frac{\pi}{4}\cos h+\cos\frac{\pi}{4}\sin h\right\}}$$

$$= \lim_{h\to 0} \frac{\left(\dfrac{1-\tan h-1-\tan h}{1-\tan h}\right)}{1-\cos h-\sin h}$$

$$= \lim_{h\to 0} \frac{-2\tan h}{(1-\tan h)(1-\cos h-\sin h)}$$

$$= \lim_{h\to 0} \frac{-2\sin h}{(\cos h-\sin h)-(\cos^2 h-\sin^2 h)}$$

$$= \lim_{h\to 0} \frac{-2h}{(\cos h-\sin h)-\cos 2h}$$

$$= \lim_{h\to 0} \frac{-2h}{\left(2\sin\frac{h}{2}\sin\frac{3h}{2}-2\sin\frac{h}{2}\cos\frac{h}{2}\right)}$$

$$= \lim_{h\to 0} \frac{-2\frac{h}{2}\times 2}{2\sin\frac{h}{2}\left(\sin\frac{3h}{2}-\cos\frac{h}{2}\right)} = 2$$

Sol.5 $\lim\limits_{x\to 1} \left(\dfrac{p}{1-x^p}-\dfrac{q}{1-x^q}\right)$

$$= \lim_{x\to 1} \frac{p(1-x^q)-q(1-x^p)}{(1-x^p)(1-x^q)}$$

$$= \lim_{h\to 0} \frac{p(1-(1+h)^q)-q(1-(1+h)^p)}{(1-(1+h)^p)(1-(1+h)^q)}$$

$$= \lim_{h\to 0} \frac{p-q-p(1+h)^q+q(1+h)^p}{(1-(1+h)^p)(1-(1+h)^q)}$$

$$= \lim_{h\to 0} \frac{(p-q)-p\left\{1+qh+\frac{q(q-1)h^2}{2!}\right\}+q\left\{1+ph+\frac{p(p-1)}{2!}h^2\right\}}{1-(1+h)^q-(1+h)^p+(1+h)^{(p+q)}}$$

$$= \lim_{h\to 0} \frac{pqh^2}{2!}(p-q) \times \frac{1}{(pqh^2)} = \left(\frac{p-q}{2}\right)$$

Sol.6 $f(x)=\dfrac{\tan x-\sin x}{\sin^3 x} \Rightarrow \lim\limits_{x\to 0} f(x)= \lim\limits_{x\to 0} \dfrac{\tan x-\sin x}{\sin^3 x}$

$$= \lim_{x\to 0} \frac{\dfrac{\tan x-\sin x}{x^3}}{\dfrac{\sin^3 x}{x^3}} = \frac{1}{2}$$

So, first term of G.P. $= \dfrac{1}{2}$

$g(x) = \dfrac{1-\sqrt{x}}{(\cos^{-1}x)^2}$, put $\cos^{-1} x = t \Rightarrow x = \cos t$

so, $g(x) = \lim\limits_{t\to 0} \dfrac{1-\sqrt{\cos t}}{t^2}$

$$= \lim_{t\to 0} \frac{1-\cos t}{t^2(1+\sqrt{\cos t})} = \frac{1}{4} \Rightarrow r = \frac{1}{4}$$

Now, $s_\infty = \dfrac{a}{1-r} = \dfrac{1/2}{1-1/4} = \dfrac{2}{3}$

Sol.7 $\lim\limits_{x\to\infty} (x-\ell n \cosh x)$ (Put $\cos ht = \dfrac{e^t+e^{-t}}{2}$)

$$= \lim_{x\to\infty} \{x-\ell n\,(e^x+e^{-x})+\ell n2\}$$

$$= \lim_{x\to\infty} \{x-\ell n e^x-\ell n\,(1+e^{-2x})+\ell n2\}$$

$$= \left[\lim_{x\to\infty} -\frac{\ln(1+e^{-2x})}{e^{-2x}}\ x\ e^{-2x}\right] + \ell n2$$

$$= \lim_{x\to\infty} -e^{-2x}+\ell n2 \Rightarrow 0+\ell n2 = \ell n2$$

Sol.8 **(a)** $\lim\limits_{x\to\frac{1}{\sqrt{2}}} \dfrac{\cos^{-1}2x\sqrt{1-x^2}}{\left(x-\dfrac{1}{\sqrt{2}}\right)}$

put $x = \sin\theta$

$$= \lim_{\theta\to\frac{\pi}{4}} \frac{\cos^{-1}(2\sin\theta|\cos\theta|)}{\sin\theta-\dfrac{1}{\sqrt{2}}}$$

$$= \lim_{\theta\to\frac{\pi}{4}} \frac{\cos^{-1}(\sin 2\theta)}{\sin\theta-\dfrac{1}{\sqrt{2}}}$$

$$= \lim_{\theta\to\frac{\pi}{4}} \frac{\frac{\pi}{2} - \sin^{-1}(\sin 2\theta)}{\sin\theta - \frac{1}{\sqrt{2}}}$$

Now apply for separate check for RHL & LHL

RHL = put $\theta = \frac{\pi}{4} + h$ & LHL put $\theta = \frac{\pi}{4} - h$

$= \frac{-1}{\sqrt{2}}$ $= \frac{1}{\sqrt{2}}$

(b) $\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{1-\sqrt{\sin 2x}}}{\pi - 4x}$

$$\text{L.H.S.} = \lim_{h\to 0} \frac{\sqrt{1-\sqrt{\sin\left(\frac{\pi}{2}-2h\right)}}}{\pi - 4\left(\frac{\pi}{4}-h\right)} = \lim_{h\to 0}\frac{\sqrt{1-\sqrt{\cos 2h}}}{4h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1-\cos 2h}}{4h} \times \frac{1}{\sqrt{1+\sqrt{\cos 2h}}}$$

$$= \frac{1}{\sqrt{2}}\lim_{h\to 0}\frac{\sqrt{2}\sinh}{4h} = \frac{1}{4}$$

$$\text{R.H.S.} = \lim_{h\to 0}\frac{\sqrt{1-\sqrt{\sin\left(\frac{\pi}{2}+2h\right)}}}{-4h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1-\cos 2h}}{-4h} \times \frac{1}{\sqrt{1+\sqrt{\cos 2h}}}$$

$$= \lim_{h\to 0}\frac{\sqrt{2}\sinh}{-4h} \times \frac{1}{\sqrt{2}} = -\frac{1}{4}$$

L.H.L. $\neq$ R.H.L. $\Rightarrow$ Limit D.N.E.

(c) $\lim_{x\to -7}\frac{[x]^2+15[x]+56}{\sin(x+7)\sin(x+8)}$

$$= \lim_{x\to -7}\frac{([x]+7)([x]+8)}{\frac{\sin(x+7)}{(x+7)}\times(x+7)\times\sin(x+8)}$$

$$= \frac{1}{\sin 1}\lim_{x\to -7}\frac{([x]+7)([x]+8)}{(x+7)} \quad \left(\frac{0}{0}\right)$$

Replace $x \to (-7-h)$

$$= \frac{1}{\sin 1}\lim_{h\to 0}\frac{([-7-h]+7)([-7-h]+8]}{-7-h+7}$$

$$= \frac{1}{\sin 1}\lim_{h\to 0}\frac{(-8+7)(-8+8)}{-h} = 0$$

Replace $x \to (-7+h)$

$$= \frac{1}{\sin 1}\lim_{h\to 0}\frac{([-7+h]+7)([-7+h]+8)}{-7+h+7}$$

$$= \frac{1}{\sin 1}\lim_{h\to 0}\frac{(-7+7)(-7+8)}{h} = 0$$

So, L.H.L. = R.H.L. = 0

Sol.9 $\lim_{x\to\frac{3\pi}{4}}\frac{1+\sqrt[3]{\tan x}}{1-2\cos^2 x} \quad \left(\frac{0}{0}\right)$

$$= \lim_{x\to\frac{3\pi}{4}}\frac{(1+\tan^{1/3}x)(1^{2/3}-\tan^{1/3}x+\tan^{2/3}x)}{-\cos 2x(1^{2/3}-\tan^{1/3}x+\tan^{2/3}x)}$$

$$= \lim_{x\to\frac{3\pi}{4}}\frac{1+\tan x}{(-\cos 2x)[1+1+1]} = \frac{1}{3}\lim_{h\to 0}\frac{1+\frac{\tanh - 1}{1+\tanh}}{-\sin 2h}$$

$$= -\frac{1}{3}\lim_{h\to 0}\frac{1}{(\cos^2 h)(1+\tanh)} = \frac{-1}{3}$$

Sol.10 $\lim_{x\to 0}\frac{8}{x^8}\left[1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right]$

$$= \lim_{x\to 0}\frac{8}{x^8}\left[1-\cos\frac{x^2}{4}-\cos\frac{x^2}{2}\left(1-\cos\frac{x^2}{4}\right)\right]$$

$$= \lim_{x\to 0}\frac{8}{x^8}\left[\left(1-\cos\frac{x^2}{2}\right)\left(1-\cos\frac{x^2}{4}\right)\right]$$

$$= \lim_{x\to 0} 8\left[\frac{2\sin^2\frac{x^2}{4}.2\sin^2\frac{x^2}{8}}{\frac{16.x^4}{16}.\frac{x^4.64}{64}}\right]$$

$$= \lim_{x\to 0}\frac{8\times 2\times 2}{16\times 648} = \frac{1}{32}$$

Sol.11 $\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}$

$$= \lim_{h\to 0} \frac{\sqrt{2} - \left\{\frac{\cos h}{\sqrt{2}} - \frac{\sin h}{\sqrt{2}}\right\} - \left\{\frac{\cos h}{\sqrt{2}} + \frac{\sin h}{\sqrt{2}}\right\}}{16h^2}$$

$$= \lim_{h\to 0} \frac{\sqrt{2}(1-\cos h)}{16\times h^2} = \frac{1}{16\sqrt{2}}$$

Sol.12

$$\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{3}+4h\right) - 4\sin\left(\frac{\pi}{3}+3h\right) + 6\sin\left(\frac{\pi}{3}+2h\right) - 4\sin\left(\frac{\pi}{3}+h\right) + \sin\frac{\pi}{3}}{h^4}$$

$$= \lim_{h\to 0} \frac{\sin\left(\frac{\pi}{3}+4h\right) + \sin\frac{\pi}{3} - 4\left\{\sin\left(\frac{\pi}{3}+h\right) + \sin\left(\frac{\pi}{3}+3h\right)\right\} + 6\sin\left(\frac{\pi}{3}+2h\right)}{h^4}$$

$$= \lim_{h\to 0} \frac{2\sin\left(\frac{\pi}{3}+2h\right)(\cos 2h - 4\cos h + 3)}{h^4}$$

$$= \lim_{h\to 0} \sqrt{3} \left\{\frac{2\cos^2 h - 4\cos h + 2}{h^4}\right\}$$

$$= \lim_{h\to 0} \sqrt{3} \left\{\frac{(2\cos h - 2)(\cos h - 1)}{h^4}\right\}$$

$$= \sqrt{3} \lim_{h\to 0} 2 \left(\frac{\cos h - 1}{h^2}\right)^2 = \sqrt{3} \times 2 \times \frac{1}{4} = \frac{\sqrt{3}}{2}$$

Sol.13 $\lim_{x\to\infty} x^2 \left(\sqrt{\frac{x+2}{x}} - \left(\frac{x+3}{x}\right)^{1/3}\right)$

$$= \lim_{h\to 0} \frac{(1+2h)^{1/2} - (1+3h)^{1/3}}{h^2}$$

$$= \lim_{h\to 0} \frac{1+h+\frac{1}{2}\left(\frac{1}{2}-1\right)^2 2h^2 - \left\{\frac{1+h+\frac{1}{3}\left(\frac{1}{3}-1\right)^2 9h^2}{2!}\right\}}{2!}$$

$$= \lim_{h\to 0} \frac{1+h-\frac{4h^2}{8} - 1 - h + \frac{2h^2}{18}\times 9}{h^2} = \frac{1}{2}$$

Sol.14 $\lim_{x\to 0} \ell n(1 + \sin^2 x) \cot \ell n^2 (1 + x)$

$$= \lim_{x\to 0} \frac{\ell n(1+\sin^2 x)}{\sin^2 x} \times \frac{\sin^2 x}{\ell n^2(1+x)} \times \frac{x^2}{x^2} = 1$$

Sol.15(i) $\lim_{x\to\infty} \left[\frac{x^2+1}{x+1} - ax - b\right] = 0$

$$\Rightarrow \lim_{x\to\infty} \left[\frac{x^2+1-ax^2-ax-bx-b}{x+1}\right] = 0$$

$$\Rightarrow \lim_{x\to\infty} \left[\frac{x^2(1-a) - x(a+b) + (1-b)}{(x+1)}\right] = 0$$

Replace $x \to \frac{1}{t}$

$$\Rightarrow \lim_{t\to 0} \left[\frac{\frac{1-a}{t^2} - \frac{(a+b)}{t} + (1-b)}{\frac{(1+t)}{t}}\right] = 0$$

$$\Rightarrow \lim_{t\to 0} \left[\frac{(1-a) - (a+b)t + (1-b)t^2}{t(t+1)}\right] = 0$$

For existence of limit, numerator = 0 $\Rightarrow a = 1$

$$\Rightarrow \lim_{t\to 0} \frac{-(b+1)t + (1-b)t^2}{t(t+1)} = 0$$

$$\Rightarrow \lim_{t\to 0} \frac{(1-b)t - (b+1)}{(t+1)} = 0$$

$\Rightarrow -(b + 1) = 0 \quad \Rightarrow \quad b = -1$

(ii) $\lim_{x\to -\infty} \left\{\sqrt{x^2 - x + 1} - ax - b\right\} = 0$

$$\Rightarrow \lim_{x\to -\infty} [\sqrt{x^2-x+1} - ax - b] = 0$$

Replace $x \to \frac{1}{t}$

$$\Rightarrow \lim_{t\to 0^-} \left[\sqrt{\frac{1}{t^2} - \frac{1}{t} + 1} - \frac{a}{t} - b\right] = 0$$

$$\Rightarrow \lim_{t\to 0^-} \left[\frac{\sqrt{t^2-t+1}}{|t|} - \frac{a}{t} - b\right] = 0$$

$$\Rightarrow -\lim_{t\to 0^-}\left[\frac{\sqrt{t^2-t+1}}{t}+\frac{a}{t}+b\right]=0$$

$$\Rightarrow -\lim_{t\to 0^-}\left[\frac{\sqrt{t^2-t+1}+a+bt}{t}\right]=0$$

For existence of limit numerator = 0

$\Rightarrow$ a + 1= 0 a = –1

$$\Rightarrow \lim_{t\to 0^-}\frac{\sqrt{t^2-t+1}-1+bt}{t}=0$$

$$\Rightarrow b=-\lim_{t\to 0^-}\frac{\sqrt{t^2-t+1}-1}{t}$$

$$\Rightarrow b=-\lim_{t\to 0^-}\frac{t^2-t+1-1}{t(\sqrt{t^2+t+1}+1)}\quad\Rightarrow\frac{1}{2}$$

So, $a=-1\ \&\ b=\frac{1}{2}$

Sol.16 $$\lim_{x\to-\infty}\frac{(3x^4+2x^2)\sin\frac{1}{x}+|x|^3+5}{|x|^3+|x|^2+|x|+1}$$

$$=\lim_{t\to 0^-}\frac{\left(\frac{3}{|t|^4}+\frac{2}{|t|^2}\right)\sin t+\frac{1}{|t|^3}+5}{\frac{1}{|t|^3}+\frac{1}{|t|^2}+\frac{1}{|t|}+1}$$

$$=\lim_{t\to 0^-}\frac{(3+2|t|^2)\sin t-t+5t^4}{t^4-t^3+t^2-t}$$

$$=\lim_{t\to 0}\frac{(3+2t^2)\frac{\sin t}{t}-1+5t^3}{t^3-t^2+t-1}=-2$$

Sol.17 $$\ell=\lim_{n\to\infty}\sum_{r=2}^{n}\left((r+1)\sin\frac{\pi}{(r+1)}-r\sin\frac{\pi}{r}\right)$$

$\{\ell\}$ = ?

r = 2, $3\sin\frac{\pi}{3}-2\sin\frac{\pi}{2}$

r = 3, $4\sin\frac{\pi}{4}-3\sin\frac{\pi}{3}$

r = 4, $5\sin\frac{\pi}{5}-4\sin\frac{\pi}{4}$

r = 5, $6\sin\frac{\pi}{6}-5\sin\frac{\pi}{5}$

r = n, $(n+1)\frac{\sin\pi}{(n+1)}-n\sin\frac{\pi}{n}$

So, $\ell=\lim_{n\to\infty}-2\sin\frac{\pi}{2}+(n+1)\sin\frac{\pi}{(n+1)}$

$$\Rightarrow \ell=\lim_{n\to\infty}-2+(n+1)\sin\frac{\pi}{(n+1)}$$

$\Rightarrow \ell=\pi-2=1.14 \quad\Rightarrow \{\ell\}=.14$

Sol.18 $$\lim_{x\to 1}\frac{(\ell n(1+x)-\ell n2)(3.4^{x-1}-3x)}{[(7+x)^{1/3}-(1+3x)^{1/2}]\sin(x-1)}$$

Replace $x\to 1+h$

$$=\lim_{h\to 0}\frac{\ln\left(\frac{2+h}{2}\right)3(4^h-1-h)}{\left[\left\{(8+h)^{1/3}-(4+3h)^{1/2}\right\}\frac{\sinh}{h}\times h\right]}$$

$$=\lim_{h\to 0}\frac{3\ln\left(1+\frac{h}{2}\right).(4^h-1-h)}{\left\{2\left(1+\frac{h}{24}\right)-2\left(1+\frac{3h}{8}\right)\right\}\frac{h}{2}\times 2}$$

$$=\frac{3}{2}\lim_{h\to 0}\frac{4^h-1-h}{\left(\frac{h}{12}-\frac{3h}{4}\right)}=\frac{3}{2}\lim_{h\to 0}\frac{1+\frac{h\ell n4}{1!}-1-h}{\frac{h}{12}-\frac{3h}{4}}$$

$$=-\frac{9}{4}\ell n\left(\frac{4}{e}\right)$$

Sol.19 $$\lim_{x\to 0}\frac{27^x-9^x-3^x+1}{\sqrt{2}-\sqrt{1+\cos x}}$$

$$=\lim_{x\to 0}\frac{(3^x)^3-(3^x)^2-3^x+1}{\sqrt{2}-\sqrt{1+\cos x}}$$

$$=\lim_{x\to 0}\frac{(3^x-1)^2(3^x+1)}{2-1-\cos x}\times(\sqrt{2}+\sqrt{1+\cos x})$$

$$= \lim_{x\to 0} \frac{(3^x-1)^2[(3^x+1)\{\sqrt{2}+\sqrt{1+\cos x}\}]}{\frac{(1-\cos x)}{x^2}\times x^2}$$

$$= 2.2\sqrt{2}\,.2\,.(\ell n3)^2 = 8\sqrt{2}\,(\ell n3)^2$$

Sol.20 $f(x) = \begin{cases} \dfrac{x}{\sin x} & , \ x > 0 \\ 2-x & , \ x \le 0 \end{cases}$

$$g(x) = \begin{cases} x+3 & , \quad x > 1 \\ x^2-2x-2 & , \quad 1 \le x < 2 \\ x-5 & , \quad x \ge 2 \end{cases}$$

L.H.L. = $\lim_{x\to 0^-} g(f(x)) = \lim_{x\to 0^-} g(f(0^-))$

$= \lim_{x\to 0^-} g(2^+) = 2 - 5 = -3$

R.H.L. = $\lim_{x\to 0^+} g(f(x)) = \lim_{x\to 0^+} g(f(0^+))$

$= \lim_{x\to 0^+} g(1) = 1^2 - 2(1) - 2 = -3$

$\because$ LHL = RHL = –3

Hence limit exists & has value = – 3

Sol.21 **(a)** If a > 0 then $\lim_{x\to 0} \tan^{-1}\left(\dfrac{a}{x^2}\right) = \dfrac{\pi}{2}$

If a = 0 then $\lim_{x\to 0} \tan^{-1}\left(\dfrac{a}{x^2}\right) = 0$

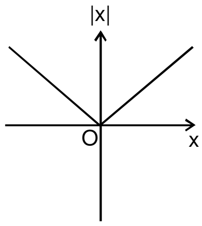
If a < 0 then $\lim_{x\to 0} \tan^{-1}\left(\dfrac{a}{x^2}\right) = \dfrac{-\pi}{2}$

(b) $f(x) = \lim_{t\to 0}\left(\dfrac{2x}{\pi} + \tan^{-1}\dfrac{x}{t^2}\right)$

If x > 0, $f(x) = \lim_{t\to 0} \dfrac{2x}{\pi} \times \dfrac{\pi}{2} = x$

If x < 0 then let x = – k

$f(x) = \lim_{t\to 0} \dfrac{-2k}{\pi}\tan^{-1}\left(\dfrac{-k}{t^2}\right)$



$= \dfrac{-2k}{\pi} \times \dfrac{-\pi}{2} = k = -x$

So, $f(x) = \begin{cases} x & , \ x \ge 0 \\ -x & , \ x < 0 \end{cases} \Rightarrow f(x) = |x|$

Sol.22 $P_n = a^{P_{n-1}} - 1 \quad \forall\ n = 2, 3, 4, ...$

$P_1 = a^x - 1$, $\quad a \in R^+$

$P_2 = a^{P_1} - 1 = a^{(a^x-1)} - 1$

$P_3 = a^{P_2} - 1 = (a^{a^{a^x-1}} - 1)$

$\Rightarrow \lim_{x\to 0} \dfrac{P_1}{x} = \lim_{x\to 0} \dfrac{a^x-1}{x} = \ell na$

$\Rightarrow \lim_{x\to 0} \dfrac{P_2}{x} = \lim_{x\to 0} \dfrac{a^{(a^x-1)}-1}{x(a^x-1)} \times (a^x - 1) = (\ell n\, a)^2$

So, proceeding same as above $\lim_{x\to 0} \dfrac{P_n}{x} = (\ell na)^n$

Sol.23 $\lim_{x\to 0} \dfrac{1}{x^3}\left(\dfrac{1}{\sqrt{1+x}} - \dfrac{1+ax}{1+bx}\right)$

$\text{Lim}_{x\to 0} \dfrac{1}{x^3}\left[\dfrac{(1+bx)-(1+ax)\sqrt{1+x}}{\sqrt{1+x}(1+bx)}\right]$

$= \text{Lim}_{x\to 0} \dfrac{(1+bx)-(1+ax)\sqrt{1+x}}{x^3}$

Now Rationalize

$= \text{Lim}_{x\to 0} \dfrac{(1+bx)^2-(1+ax)^2(1+x)}{x^3[(1+bx)+(1ax)\sqrt{1+x}]}$

$= \text{Lim}_{x\to 0} \dfrac{-a^2x^3 + x^2[b^2-a^2-2a] + x(2b-2a-1)}{2x^3}$

For Existance of limit

$b^2 - a^2 - 2a = 0 \quad \& \ 2b - 2a - 1 = 0$

$a = 1/4 \qquad b = 3/4$

$\ell = \dfrac{-a^2}{2} = \dfrac{-1}{32}$

$\Rightarrow \dfrac{1}{a} - \dfrac{2}{\ell} + \dfrac{3}{b} = 72$

Sol.24 $\lim_{y\to 0}\left[\lim_{x\to\infty} \dfrac{e^{x\ell n\left(1+\frac{ay}{x}\right)} - e^{x\ell n\left(1+\frac{by}{x}\right)}}{y}\right]$

$$= \lim_{y\to 0}\left[\frac{\lim_{x\to\infty}\dfrac{\ell n\left(1+\frac{ay}{x}\right)}{\left(\frac{1}{x}\right)}-\dfrac{\ell n\left(1+\frac{by}{x}\right)}{\left(\frac{1}{x}\right)}}{y}\right]$$

$$= \lim_{y\to 0}\frac{\left[\lim_{x\to\infty}\left\{\dfrac{2\ln(1+ay/x)}{\frac{ay}{x}}\right\}\times ay-\left\{\dfrac{\ln(1+by/3)}{(by/x)}\right\}by\right]}{y}$$

$$= \lim_{y\to 0}\frac{ay-by}{y} = (a-b)$$

Sol.25 **(i)** $a_n + b_n + c_n = 2n + 1$

we can define one cubic whose roots are a_n, b_n, c_n

$x^3 - (2n+1)x^2 + (2n-1)x + 1 = 0$

$(x-1)(x^2 - 2nx - 1) = 0$

$x = 1,\ n \pm \sqrt{n^2+1}$

$a_n = n - \sqrt{n^2+1}$

$\lim\limits_{h\to\infty} n\,(n - \sqrt{n^2+1}) = -1/2$

Sol.26 $a_n = 2^2\,[1^2 + 2^2 + 3^2 + \ldots.. + n^2]$

$$= \frac{4n(n+1)(2n+1)}{6}$$

$b_n = \Sigma(2n)^2 - 4\,\Sigma n^2$

$$= \frac{2n(2n+1)(4n+1)}{6} - \frac{4n(n+1)(2n+1)}{6}$$

Hence $\lim\limits_{n\to\infty}\dfrac{\sqrt{a_n}-\sqrt{b_n}}{\sqrt{n}} = \dfrac{\sqrt{3}}{2}$

Sol.27 $\lim\limits_{x\to\infty}\left[\dfrac{2x^2+3}{2x^2+5}\right]^{8x^2+3}$

$\Rightarrow \ell = \lim\limits_{x\to\infty}(8x^2+3)\left\{\dfrac{-2}{2x^2+5}\right\} = \dfrac{-2\times 8}{2} = -8$

So, $e^{\ell} = e^{-8}$

Sol.28 $\lim\limits_{x\to\infty}\left(\dfrac{x+c}{x-c}\right)^x = 4$

$\Rightarrow \quad e^{\ell} = e^{\lim\limits_{x\to\infty} x\left(\frac{2c}{x-c}\right)} = 4 \ \Rightarrow e^{\lim\limits_{x\to\infty} 2c} = 4$

$\Rightarrow \ e^{2\ell} = 4 \qquad \Rightarrow \qquad e^{2c} = e^{\ell n4} = 2c = \ell n4$

$\Rightarrow \qquad c = \dfrac{\ell n4}{2} = \ell n\,2$

Sol.29 $\lim\limits_{x\to 1}\left\{\tan\left(\dfrac{\pi x}{4}\right)\right\}^{\tan\left(\frac{\pi x}{2}\right)}$

$\Rightarrow \ \ell = \lim\limits_{x\to 1}\tan\dfrac{\pi x}{2}\left(\tan\left(\dfrac{\pi x}{2}\right)-1\right)$

Replace $x \to 1 + h$

$\Rightarrow \ell = \lim\limits_{h\to 0}\tan\left(\dfrac{\pi}{2}+\dfrac{\pi h}{2}\right)\left\{\tan\left(\dfrac{\pi}{4}+\dfrac{\pi h}{4}\right)-1\right\}$

$\Rightarrow \ell = \lim\limits_{h\to 0} -\cot\dfrac{\pi h}{2}\left\{\dfrac{1+\tan\frac{\pi h}{4}}{1-\tan\frac{\pi h}{4}}-1\right\}$

$\Rightarrow \ell = \lim\limits_{h\to 0} -\cot\dfrac{\pi h}{2}\left\{\dfrac{2\tan\frac{\pi h}{4}}{1-\tan\frac{\pi h}{4}}\right\}$

$\Rightarrow \ell = \lim\limits_{h\to 0} -\dfrac{-\cos(\pi h/2)}{\sin\left(\frac{\pi h}{2}\right)}\left\{\dfrac{2\sin\frac{\pi h}{4}}{\cos\frac{\pi h}{4}-\sin\frac{\pi h}{4}}\right\}$

$\Rightarrow \ell = \lim\limits_{h\to 0}\dfrac{-\cos(\pi h/2)}{2\sin(\pi h/4)\cos\frac{\pi h}{4}} \times \dfrac{2\sin\frac{\pi h}{4}}{\cos\frac{\pi h}{4}-\sin\frac{\pi h}{4}}$

$\Rightarrow -\dfrac{1\times 1}{1} = -1 \qquad \Rightarrow \qquad e^{-1}$

Sol.30

EXERCISE – IV

HINT & SOLUTION

Sol.1 $\lim_{x\to\infty} x^2 \sin \ell n\sqrt{\cos\frac{\pi}{x}}$

Put x= $\frac{1}{h}$ so, $\lim_{h\to 0}\frac{\sin\left(\ell n\sqrt{\cos\pi h}\right)}{h^2}$

$= \lim_{h\to 0}\frac{\sin(\ell n\sqrt{\cos\pi h})}{\ell n\sqrt{\cos\pi h}}\times\frac{\ell n\sqrt{\cos\pi h}}{h^2}$

$= \lim_{h\to 0}\frac{\ln\left(1+\sqrt{\cos\pi h}-1\right)}{h^2}$

$= \lim_{h\to 0}\frac{\ell n\left[1+\left\{-\left(1-\sqrt{\cos\pi h}\right)\right\}\right]}{-\left(1-\sqrt{\cos\pi h}\right)}\times\frac{-\left(1-\sqrt{\cos\pi h}\right)}{h^2}$

$= \lim_{h\to 0}\frac{-(1-\cos\pi h)}{\pi^2h^2}\times\frac{\pi^2}{\left(1+\sqrt{\cos\pi h}\right)}$

$= -\frac{\pi^2}{2\times 2} = -\frac{\pi^2}{4}$

Sol.2 $\lim_{x\to\infty}\left[\cos\left\{2\pi\left(\frac{x}{1+x}\right)^a\right\}\right]^{x^2}$, $a\in Q$

$= \lim_{x\to\infty}\left[\cos 2\pi\left(\frac{1}{1+\frac{1}{x}}\right)^a\right]^{x^2}$

$= \lim_{x\to\infty}\left\{\cos 2\pi\left(1+\frac{1}{x}\right)^{-a}\right\}^{x^2}$ $\quad (1^\infty)$

$\ell = \lim_{x\to\infty} x^2\left\{\cos 2\pi\left(1+\frac{1}{x}\right)^{-a}-1\right\}$

$= \lim_{x\to\infty} x^2\frac{\left\{\cos\left(\frac{2\pi a}{x}\right)-1\right\}}{\left(\frac{2\pi a}{x}\right)^2}\times\left(\frac{2\pi a}{x}\right)^2$

so $e^{\ell} = e^{-2\pi^2a^2}$

Sol.3 $F(x)=\frac{\sin^{-1}\left(1-\{x\}\right).\ \cos^{-1}\left(1-\{x\}\right)}{\sqrt{2\{x\}}\left(1-\{x\}\right)}$

R.H.L. = $\lim_{x\to 0^+} f(x)$

$= \lim_{x\to 0^+}\frac{\sin^{-1}\left(1-\{x\}\right)}{\left(1-\{x\}\right)}\times\frac{\sin^{-1}\sqrt{2\{x\}-\{x\}^2}}{\sqrt{2\{x\}}}$

$= \lim_{x\to 0^+}\left[\frac{\sin^{-1}\left(1-\{x\}\right)}{1-\{x\}}\right]\times\frac{\sqrt{2\{x\}-\{x\}^2}}{\sqrt{2\{x\}}}$

$= \lim_{x\to 0^+}\frac{\pi}{2}\times\sqrt{1-\frac{\{x\}}{2}} = \frac{\pi}{2}$

LHL = $\lim_{x\to 0^-}\frac{\sin^{-1}\left(1-\{x\}\right).\cos^{-1}\left(1-\{x\}\right)}{\sqrt{2\{x\}}\left(1-\{x\}\right)}$

$= \lim_{x\to 0^-}\left[\frac{\sin^{-1}\left(1-\{x\}\right)}{\left(1-\{x\}\right)}\right]\times\frac{\sin^{-1}\sqrt{2\{x\}-\{x\}^2}}{\sqrt{2\{x\}}}$

$= 1\times\frac{\pi}{2\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$

Sol.4 $\lim_{n\to\infty}\left(\frac{\sqrt{n^2+n}-1}{n}\right)^{2\sqrt{n^2+n}-1}$

since it is of the form 1^∞, so,

$\ell = \lim_{n\to\infty}\left(2\sqrt{n^2+n}-1\right)\left\{\frac{\sqrt{n^2+n}-1-n}{n}\right\}$

$= \ell = \lim_{n\to\infty}\frac{2\left(n^2+n\right)-2\sqrt{n^2+n}-2n\sqrt{n^2+n}-\sqrt{n^2+n}+1+n}{n}$

$= \lim_{n\to\infty}\left[\frac{2n^2+3n-\sqrt{n^2+n}\,(2n+3)}{n}+\frac{1}{n}\right] = -1$

so, e^{-1}

Sol.5 $\lim_{x\to\infty}\left(\frac{a_1^{\frac{1}{x}}+a_2^{\frac{1}{x}}+....+a_n^{\frac{1}{x}}}{n}\right)^{nx}$

It is 1^∞ form so,

$$\ell = \lim_{x\to\infty} nx\left[\frac{a_1^{\frac{1}{x}} + a_2^{\frac{1}{x}} + \ldots + a_n^{\frac{1}{x}} - n}{n}\right]$$

replace $x \to \frac{1}{h}$

$$= \lim_{h\to 0}\left[\frac{(a_1^h - 1)}{h} + \frac{(a_2^h - 1)}{h} + \ldots + \frac{a_n^h - 1}{h}\right]$$

$= \ell n a_1 + \ell n a_2 + \ldots + \ell n a_n$

$= \ell n(a_1.a_2.....a_n)$

so, given limit = $e^{\ell} = e^{\ell n(a_1.a_1.....a_n)}$

= $(a_1.\ a_2......a_n)$

Sol.6 $\lim_{x\to 0}\left[\frac{(1+x)^{\frac{1}{x}}}{e}\right]^{\frac{1}{x}}$ (1^∞)

$$\ell = \lim_{x\to 0}\frac{1}{x}\left\{\frac{\frac{\ell n(1+x)}{x} - e}{e}\right\} = \lim_{x\to 0}\frac{1}{x}\left\{\frac{e^{\left(\frac{x-\frac{x^2}{2}}{x}\right)} - e}{e}\right\}$$

$$= \lim_{x\to 0}\frac{1}{x}\left\{\frac{e^{\left(1-\frac{x}{2}\right)} - e}{e}\right\} = \lim_{x\to 0}\frac{1}{x}\left\{e^{-\frac{x}{2}} - 1\right\}$$

$$= \lim_{x\to 0}\frac{1}{x}\left\{1 - \frac{x}{2} - 1\right\} = -\frac{1}{2}$$

so, given limit = $e^{\ell} = e^{-1/2}$

Sol.7 $\lim_{x\to\infty}\left[\frac{\cosh(\pi/x)}{\cos(\pi/x)}\right]^{x^2}$ (1^∞)

$$\ell = \lim_{x\to\infty} x^2\left[\frac{e^{\frac{\pi}{x}} + e^{\frac{-\pi}{x}}}{2\cos\left(\frac{\pi}{x}\right)} - 1\right]$$

$$= \lim_{x\to\infty} x^2\left[\frac{e^{\frac{\pi}{x}} + e^{\frac{-\pi}{x}} - 2\cos\left(\frac{\pi}{x}\right)}{2\cos\left(\frac{\pi}{x}\right)}\right]$$

$$= \lim_{x\to\infty} x^2\left[\frac{1 + \frac{\pi}{x} + \frac{\pi^2}{x^2\,2!} + 1 - \frac{\pi}{x} + \frac{\pi^2}{x^2\,2!} - 2\left(1 - \frac{\pi^2}{x^2\,2!}\right)}{2 - \frac{\pi^2}{x^2}}\right]$$

$= \pi^2 \qquad \Rightarrow \qquad$ limit = $e^{\ell} = e^{\pi^2}$

Sol.8 $\lim_{x\to a}\frac{1}{(a+x)^2(a-x)^2}\left[\frac{a^2+x^2}{ax} - 2\sin\left(\frac{a\pi}{2}\right)\sin\left(\frac{\pi x}{2}\right)\right]$

$$= \frac{1}{4a^2}\lim_{x\to a}\frac{\left(\frac{a}{x} + \frac{x}{a}\right) - 2\sin\left(\frac{a\pi}{2}\right)\sin\left(\frac{\pi x}{2}\right)}{(a-x)^2}$$

Applying L' Hosp. rule & putting limit

$$= \frac{1}{4a^2}\times\frac{1}{2}\left(\frac{2a}{a^3} + \frac{\pi^2}{2}\sin^2\left(\frac{a\pi}{2}\right)\right)$$

$$= \frac{1}{8a^2}\left(\frac{2}{a^2} + \frac{\pi^2}{2}(1)^2\right) = \frac{1}{16a^4}\left(4 + \pi^2 a^2\right)$$

Sol.9 $L = \lim_{x\to 1}\frac{(1-x)(1-x^2).....(1-x^{2n})}{\left[(1-x)(1-x^2).....(1-x^n)\right]^2}$

$$= \lim_{x\to 1}\frac{\left(\frac{1-x}{1-x}\right).\left(\frac{1-x^2}{1-x}\right)......\left(\frac{1-x^{2n}}{1-x}\right)}{\left[\left(\frac{1-x}{1-x}\right)\left(\frac{1-x^2}{1-x}\right).....\left(\frac{1-x^n}{1-x}\right)\right]^2}$$

$$\frac{1.2\ \ldots\ldots\ldots\ldots 2n}{(1.2\ \ldots\ldots\ldots\ldots n)^2} = \frac{2n!}{n!.n!} = \frac{2}{n!} = {}^{2n}C_n$$

(a) $\prod_{r=1}^{n}\frac{n+r}{r} \Rightarrow (n+1)\left(\frac{n+2}{2}\right) + \ldots + \frac{n+n}{n}$

$$= \frac{n!\left[(n+1)(n+2).....2n\right]}{n!\,n!} = \frac{2n!}{n!.n!} = {}^{2n}C_n$$

(b) $\frac{1}{n!}\prod_{r=1}^{n}(4r-2) \Rightarrow \frac{2^n}{n!}\prod_{r=1}^{n}(2r-1)$

$$= \frac{2^n}{n!}\frac{1.3.5.....(2n-1)(2.4.....2n)}{(2.4.6.8.....2n)} = {}^{2n}C_n$$

(c) $T_{r+1} = {}^{2n-1}C_r\,(1).\,x^{2n-1}$

No. of total term 2n

1st middle term $\frac{2n}{2} = n$

2nd middle term $\frac{2n}{2} + 1 = n + 1$

coefficient of n^{th} term

$T_n = T_{n-1+1} = {}^{2n-1}C_{n-1}$

$T_{n+1} = {}^{2n-1}C_n$

$\Rightarrow \quad T_n + T_{n+1} = {}^{2n-1}C_{n-1} + {}^{2n-1}C_n$

$$= \frac{(2n-1)!}{(n-1)!(2n-1-n+1)!} + \frac{(2n-1)!}{n!(2n-1-n)!}$$

$$= \frac{2n!}{(n!)^2} = 2n$$

(d) The coefficient of x^n in $(1+x)^{2n}$

$T_{r+1} = {}^{2n}C_r(1)\,x^r$

$T_{n+1} = {}^{2n}C_n\,x^n = \frac{2n!}{(n!)^2}$

Sol.10 $\lim\limits_{x\to\infty} \frac{x^3(2a+b-3c) + x^2(5b-a-c) - b}{x^4(5a-b+4c) + 2x^2 + x(5-a) + c} = 1$

5a – b + 4c = 0 ....(1)

2a + b – 3c = 0 ....(2)

5b – a – c = 2 ....(3)

$a = \frac{-b}{c} = 1$

Solving for a, b, c we get $a = \frac{-2}{109}$, $b = \frac{46}{109}$, $c = \frac{14}{109}$

$\Rightarrow a+b+c = \frac{58}{109} = \frac{p}{q} \Rightarrow p+q = 58 + 109 = 167$

Sol.11

Sol.12 $\lim\limits_{x\to\infty}\left[\frac{\ell n(1+x)^{1+x}}{x^2} - \frac{1}{x}\right]$

$$\lim_{x\to\infty}\left[\frac{(1+x)\left(x - \frac{x^2}{2} + \frac{x^3}{3}.....\right) - x}{x^2}\right]$$

$$\lim_{x\to\infty}\frac{\left[\left(1 - \frac{x}{2} + \frac{x^2}{2} + \frac{x^3}{3}....\right) + \left(x - \frac{x^2}{2} + ...\right) - 1\right]}{x^2} = \frac{1}{2}$$

Sol.13 Let $L = \prod\limits_{n=3}^{\infty}\left(1 - \frac{4}{n^2}\right), M = \prod\limits_{n=2}^{\infty}\frac{n^3-1}{n^3+1}, N = \prod\limits_{n=1}^{\infty}\frac{(1+n^{-1})^2}{(1+2n^{-1})}$

$$L = \prod_{n=3}^{\infty}\frac{(n+2)(n-2)}{n.n} = \frac{1}{3}\frac{2}{4}\frac{3}{5}....\left(\frac{n-2}{n}\right)\frac{5}{3}\frac{6}{4}\frac{7}{5}...\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)$$

$$= \frac{1.2.(n+1)(n+2)}{n(n-1).3.4} = \frac{1}{6}$$

$$M = \prod_{n=2}^{\infty}\left(\frac{n^3-1}{n^3+1}\right) = \frac{1}{3}\frac{2}{4}\frac{3}{n}.....\left(\frac{n-1}{n+1}\right).\frac{7}{3}.\frac{13}{7}....\left(\frac{n^2+n+1}{n^2-n+1}\right) = \frac{2}{3}$$

$$N = \prod_{n=1}^{\infty}\frac{\left(1 + \frac{1}{n}\right)^2}{\left(1 + \frac{2}{n}\right)} \Rightarrow \prod_{n=1}^{\infty}\frac{(n+1)^2}{n^2\left(\frac{n+2}{n}\right)}$$

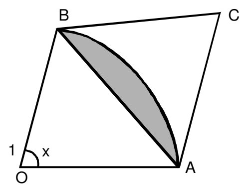
$$= \lim_{n\to\infty}\frac{2}{1}.\frac{3}{2}...\frac{n+1}{n}...\frac{2}{3}.\frac{3}{4}.....\left(\frac{n+1}{n+2}\right)$$

$$= \lim_{n\to\infty}\frac{2n\left(1 + \frac{1}{n}\right)}{n\left(1 + \frac{2}{n}\right)} = 2$$

$$\Rightarrow \quad L^{-1} + M^{-1} + N^{-1} = \left(\frac{1}{6}\right)^{-1} + \left(\frac{2}{3}\right)^{-1} + 2^{-1} = 8$$

Sol.14

(a) $\ell n\ \Delta OBC$,



$\tan\frac{x}{2} =$ BC = CA,

$\angle$ BCA = $\pi - x$

$$\therefore T(x) = \frac{1}{2}\times CB.CA.\ \sin(\pi - x)$$

$$= \frac{1}{2}\tan^2\left(\frac{x}{2}\right)\sin x = \tan\frac{x}{2} - \sin\frac{x}{2}$$

(b) S(x) = Area of sector OBA – Area of ΔOBA

$$= \frac{1}{2}(1)^2 x - \frac{1}{2}\sin x = \frac{x - \sin x}{2}$$

(c) $\lim\limits_{x\to 0}\frac{T(x)}{S(x)} = \lim\limits_{x\to 0}\frac{\tan\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)}{\left(\frac{x - \sin x}{2}\right)}$

$$= \lim_{x \to 0} \frac{\tan^2\left(\frac{x}{2}\right)\sin x}{(x - \sin x)}$$

$$= \lim_{x \to o} \frac{\frac{(\tan(x/2))^2}{(x/2)^2} \cdot \frac{\sin x}{x}}{\left(\frac{x - \sin x}{x^3}\right)} \cdot \frac{1}{4} = 3/2$$

Sol.15 $f(x) = \lim_{n \to \infty} \frac{1}{4} \sum_{r=1}^{n} 3^{r-1}\left\{3\sin\frac{x}{3^r} - \sin\left(\frac{3x}{3^r}\right)\right\}$

$$= \frac{1}{4} \lim_{n \to \infty}\left\{3\sin\frac{x}{3} - \sin x + 3^2 \sin\frac{x}{3^2} - 3\sin\frac{x}{3} \ldots - 3^{n-1}\sin\frac{x}{3^{n-1}}\right\}$$

$$= \frac{1}{4} \lim_{n \to \infty}\left(-\sin x + 3^n \sin\frac{x}{3^n}\right)$$

$$= \frac{-\sin x}{4} + \lim_{n \to \infty} \frac{3^{n-1}}{4} \cdot \frac{\sin\left(\frac{x}{3^n}\right)}{\left(\frac{x}{3^n}\right)} \times \left(\frac{x}{3^n}\right)$$

$$= \frac{-\sin x}{4} + \frac{x}{4}$$

Now, $g(x) = x - 4f(x)$

$g(x) = x + \sin x - x = \sin x$

Now, $\lim_{x \to 0} (1 + \sin x)^{\cot x} = e$

Sol.16 $f(n, \theta) \prod_{r=1}^{n} \left\{1 - \tan^2\left(\frac{\theta}{2^r}\right)\right\}$

we know that, $\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$

$$\Rightarrow (1 - \tan^2\theta) = \frac{2\tan\theta}{\tan 2\theta}$$

so, $f(n,\theta) = \prod_{r=1}^{n} \frac{2\tan\left(\frac{\theta}{2r}\right)}{\tan\left(\frac{\theta}{2^{r-1}}\right)}$

$$= \frac{2\tan\frac{\theta}{2} \cdot 2\tan\frac{\theta}{2^2} \ldots\ldots 2\tan\frac{\theta}{2^n}}{\tan\theta \cdot \tan\frac{\theta}{2} \cdot \tan\frac{\theta}{2^2} \ldots\ldots \tan\frac{\theta}{2^{n-1}}} = \frac{2^n \tan\left(\frac{\theta}{2^n}\right)}{\tan\theta}$$

$$\lim_{n \to \infty} f(n,\theta) = \lim_{n \to \infty} \frac{2^n \tan\left(\frac{\theta}{2^n}\right)}{\tan\theta} = \frac{\theta}{\tan\theta}$$

Sol.17 $L = \lim_{x \to 0} \frac{\frac{1}{\sqrt{2}}\sqrt{\cos 2x + (1+3x)^{1/3}}\ \sqrt[3]{\cos^3 x - \ell og\,(1+x)}}{x}$

$$L = \lim_{x \to 0} \frac{1}{\sqrt{2}} \frac{\left[-2\sin 2x + \frac{3}{3}(1+3x)^{-2/3}\right]}{\sqrt{\cos 2x + (1+3x)^{1/3}}}$$

$$\frac{1}{3}\left[\cos^3 x - \log(1+x)\right]\left[-3\cos^2 x \sin x - \frac{1}{1+x}\right] = \frac{7}{12}$$

$$L = \frac{a}{b} = \frac{7}{12} \Rightarrow a + b = 19$$

Sol.18 $\lim_{x \to 0} \frac{x(1 + a\cos x) - b\sin x}{\frac{\{f(x)\}^3}{x^3} \times x^3}$

$$\lim_{x \to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3}$$

$$\lim_{x \to 0} \frac{x\left(1 + a - \frac{ax^2}{2!} + \frac{ax^4}{4!}\right) .. b\left(x - \frac{x^3}{3!} + \frac{x^5}{x!} \ldots\right)}{x^3}$$

$$\lim_{x \to 0} \frac{x(1 + a - b) + x^3\left(\frac{b}{6} - \frac{a}{2}\right) + \ldots}{x^3}$$

$1 + a - b = 0 \quad \Rightarrow \quad b = a + 1$

$\frac{b}{6} - \frac{a}{2} = 1 \quad \Rightarrow \quad a = -\frac{5}{2}\ \ b = -\frac{3}{2}$

Sol.19 Apply Geometric Condition & proceed as in question number 21

Sol.20 Apply Geometric Condition & proceed as in question number 21

Sol.21 Δ_1 = Area of $\Delta ABC = R^2 \sin\theta\,(\sec\theta - \cos\theta)$

$\Delta_1 = R^2 \tan\theta\,(1 - \cos^2\theta)$

$$\text{Area of } \Delta CDE = \frac{R^2(1 - \cos\theta)^2}{\cos^2\theta, \tan\theta}$$

$$= \Delta_2 \begin{bmatrix} CM = R\sec\theta - R \\ DM = cm\cos\theta \end{bmatrix}$$

$$\frac{\Delta_1}{\Delta_2} = \frac{\tan\theta(1-\cos^2\theta)\cos^2\theta\tan\theta}{(1-\cos\theta)^2} = L\text{ (say)}$$

$$\Rightarrow \lim_{\theta\to 0} L = \lim_{\theta\to 0} \frac{\tan^2\theta\cos^2\theta)(1-\cos\theta)(1+\cos\theta)}{(1-\cos\theta)^2}$$

$$= 2\lim_{\theta\to 0} \frac{\tan^2\theta}{\theta^2} \times \frac{\theta^2}{1-\cos\theta}$$

$$= 4$$

Sol.22 $L = \lim_{n\to 0} \frac{\ln\left(x+\sqrt{1+x^2}\right)-\ln(1+x)}{\ln(1+x)\,.\,\ln\left(n+\sqrt{1+x^2}\right)}$

Applying L′pital Rule : $L = \frac{1}{2}$

Now, $\frac{L+153}{L} = 1+306 = 307$

Sol.23 **(a)** $\lim_{x\to\infty} x\,f(x) = \lim_{x\to\infty} 2x\sin\frac{1}{x} = 2$

(b) $\lim_{x\to 1} f(x)$

$\text{RHL} = \lim_{x\to 1^+} f(x) = \lim_{h\to 0} 2\sin\left(\frac{1}{1+h}\right) = 2\sin 1$

$\text{LHL} = \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1-h) = 1$

(c) $\lim_{x\to 0} f(x) = \lim_{x\to 0} x = 0$

(d) $\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \sin\frac{1}{x} = 0$

Sol.24 **(a)** $\ell = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+n}}+\ldots+n\right) \le \ell \lim_{n\to\infty} \frac{1}{\sqrt{n^2}}+\ldots.+\frac{1}{\sqrt{n^2}}$

$$\lim_{n\to\infty}\frac{2n+1}{\sqrt{n^2+2n}} \le \ell \le \lim_{n\to\infty}\frac{2n}{\sqrt{n^2}}$$

$$2 \le \ell \le 2 \quad \Rightarrow \ell = 2$$

(b) $\lim_{n\to\infty}\left(\frac{1}{1+n^2}+\frac{2}{2+n^2}+\ldots.+\frac{n}{n+n^2}\right)$

$$L = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{r^2+n^2}$$

$$\frac{1}{n+n^2} \le \frac{1}{1+n^2} \le \frac{1}{1+n^2}$$

$$\Rightarrow \frac{2}{n+n^2} \le \frac{2}{2+n^2} \le \frac{2}{1+n^2}$$

$$\lim_{n\to\infty}\left(\frac{1}{n+n^2}+\frac{2}{n+n^2}+\ldots+\frac{n}{n+n^2}\right) \le L \le \lim_{n\to\infty}\left(\frac{1}{1+n^2}+\ldots.\frac{n}{1+n^2}\right)$$

$$\frac{1}{2} \le \ell \le \frac{1}{2} \Rightarrow \ell = \frac{1}{2}$$

Sol.25

EXERCISE – V HINT & SOLUTION

Sol.1 C

For $x \in R$, $\lim_{x \to \infty}\left(\frac{x-3}{x+2}\right)^x = ?$

The format is 1^∞, so , given limit = e^ℓ

$\ell = \lim_{x \to \infty} x\left(\frac{x-3}{x+2}-1\right) = \lim_{x \to \infty} x\left(\frac{-5}{x+2}\right)$ = -5

so, e^{-5}

Sol.2 B

$$\ell = \lim_{x \to 0} \frac{\sin\left(\pi\cos^2 x\right)}{x^2}$$

$$= \lim_{x \to 0}\frac{\sin\left(\pi - \pi\sin^2 x\right)}{x^2} = \lim_{x \to 0}\frac{\sin\left(\pi\sin^2 x\right)}{x^2}$$

$$\ell = \lim_{x \to 0}\frac{\sin\left(\pi\sin^2 x\right)}{\pi\left(\sin^2 x\right)} \times \frac{\pi\sin^2 x}{x^2} = \pi$$

Sol.3 $\ell = \lim_{x \to 0}\frac{a^{\sin x}\left(a^{\tan x - \sin x} - 1\right)}{\left(\tan x - \sin x\right)}$ = $a^0\ \ell n\ a = \ell na$

Sol.4 C

$$\ell = \lim_{x \to 0}\frac{(\cos x - 1)}{x^2}\frac{\left\{\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+..\right)-\left(1+x+\frac{x^2}{2!}+..\right)\right\}}{x^1}$$

so clearly the value of n must be '3'

Sol.5 C

$$\ell = \lim_{x \to 0}\frac{\sin(nx)\left[(a-n)nx - \tan x\right] = 0}{x^2}$$

$$\ell = \lim_{x \to 0}\frac{\sin nx}{nx}\times n\left[\frac{(a-n)nx}{x} - \frac{\tan x}{x}\right] = 0$$

$$\ell = \lim_{x \to 0} 1 \times n\left[an - n^2 - 1\right] = 0 \Rightarrow a = \frac{n^2+1}{n}$$

Sol.6 $\ell = \lim_{n \to \infty}\left[\frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right) - n\right]$

Put $n = \frac{1}{h}$

$$\ell = \lim_{h \to 0}\left[\frac{2}{\pi h}(1+h)\cos^{-1}h - \frac{1}{h}\right]$$

$$\ell = \lim_{h \to 0}\left(\frac{(2+2h)\cos^{-1}h - \pi}{\pi h}\right)$$

$$\ell = \lim_{h \to 0}\frac{\frac{-(2+2h)}{\sqrt{1-h^2}} + 2\cos^{-1}h - 0}{\pi}$$

$$\ell = \frac{\pi - 2}{\pi} = \left(1 - \frac{2}{\pi}\right)$$

Sol.7 A,C

$$\ell = \mathrm{Lim}_{x \to 0}\frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}$$

$$\ell = \lim_{x \to 0}\frac{a - \left(a^2 - x^2\right)^{1/2} - \frac{x^2}{4}}{x^4}$$

$$\ell = \lim_{x \to 0}\frac{a - a\left\{1 - \frac{x^2}{2a^2} + \frac{1}{2}\times\left(\frac{-1}{2}\right)\frac{x^4}{2a^2}\right\} - \frac{x^2}{4}}{x^4}$$

$$\ell = \frac{\left(\frac{ax^2}{2a^2} - \frac{x^2}{4}\right) + \frac{ax^4}{8a^4}}{x^4}$$

Far 'L' to the finite, $\frac{x^2}{2a} - \frac{x^2}{4} = 0$

a = 2, L = $\frac{1}{8a^3} = \frac{1}{64}$

Answer Ex–I SINGLE CORRECT (OBJECTIVE QUESTIONS)

1. C **2.** D **3.** D **4.** B **5.** D **6.** C **7.** C
8. C **9.** A **10.** C **11.** B **12.** C **13.** A **14.** A
15. B **16.** C **17.** C **18.** D **19.** D **20.** B **21.** C
22. C **23.** B **24.** C **25.** A **26.** B **27.** A **28.** B
29. C **30.** C **31.** A **32.** D **33.** C **34.** A **35.** B
36. B **37.** B **38.** A **39.** A **40.** D **41.** C **42.** A
43. B **44.** C **45.** C **46.** A **47.** C **48.** D **49.** A
50. C **51.** B **52.** A **53.** D **54.** A **55.** A **56.** B
57. A **58.** C

Answer Ex–II MULTIPLE CORRECT (OBJECTIVE QUESTIONS)

1. A,B,C **2.** A,B **3.** A,B **4.** A,B,C,D **5.** A,D **6.** A,D **7.** B,C,D
8. A,B,C,D **9.** A,C **10.** B,D

Answer Ex–III SUBJECTIVE QUESTIONS

1. $\frac{45}{91}$ **2.** 2 **3.** 5050 **4.** 2 **5.** $\frac{p-q}{2}$ **6.** $a = \frac{1}{2}$; $r = \frac{1}{4}$; $S = \frac{2}{3}$

7. $\ell n\ 2$ **8.** **(a)** does not exist ; **(b)** does not exist ; **(c)** 0 **9.** $-\frac{1}{3}$ **10.** $\frac{1}{32}$ **11.** $\frac{1}{16\sqrt{2}}$

12. $\frac{\sqrt{3}}{2}$ **13.** 1/2 **14.** 1 **15.** **(i)** a = 1, b = –1 **(ii)** a = –1, b = $\frac{1}{2}$ **16.** –2 **17.** $\pi - 3$

18. $-\frac{9}{4}\ \ell n\ \frac{4}{e}$ **19.** $8\sqrt{2}\,(\ell n\ 3)^2$ **20.** –3, –3, –3 **21.** (a) $\pi/2$ if a>0; 0 if a = 0 and $-\pi/2$ if a < 0; (b) f(x) = | x |

22. $(\ell n\ a)^n$ **23.** 72 **24.** a – b **25.** –1/2 **26.** $\frac{\sqrt{3}}{2}$ **27.** e^{-8} **28.** c = ℓn 2 **29.** e^{-1} **30.** $e^{-1/2}$

Answer Ex–IV ADVANCED SUBJECTIVE QUESTIONS

1. $-\frac{\pi^2}{4}$ **2.** $e^{-2\pi^2 a^2}$ **3.** $\frac{\pi}{2}, \frac{\pi}{2\sqrt{2}}$ **4.** e^{-1} **5.** $(a_1.a_2.a_3 ......a_n)$ **6.** $e^{-\frac{1}{2}}$

7. e^{π^2} **8.** $\frac{\pi^2 a^2 + 4}{16a^4}$ **10.** 167 **11.** $\frac{\pi}{3}$ **12.** 1/2 **13.** 8

14. $T(x) = \frac{1}{2}\tan^2\frac{x}{2}.\sin x$ or $\tan\frac{x}{2} - \frac{\sin x}{2}$, $S(x) = \frac{1}{2}x - \frac{1}{2}\sin x$, limit = $\frac{3}{2}$ **15.** g(x) = sin x and ℓ = e

16. $\frac{\theta}{\tan\theta}$ **17.** 19 **18.** a = – 5/2, b = –3/2 **20.** $\frac{2L}{3}$ **21.** 4 **22.** 307
23. (a) 2, (b) D.N.E., (c) 0, (d) 0 **24.** (a) 2 ; (b) 1/2 **25.** 683

Answer Ex–V JEE PROBLEMS

1. C **2.** B **3.** ℓn a **4.** C **5.** C **6.** $1 - \frac{2}{\pi}$ **7.** A, C